Evolution of magnetic field fluctuations in chaotic flows.

I. INTRODUCTION

The evolution equation of the field $\boldsymbol{B}(\boldsymbol{r},t)$ in the incompressible flow $\boldsymbol{v}(\boldsymbol{r},t)$ has the form :

$$\partial_t \boldsymbol{B} = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{B} + \kappa\nabla^2\boldsymbol{B}. \qquad (1)$$

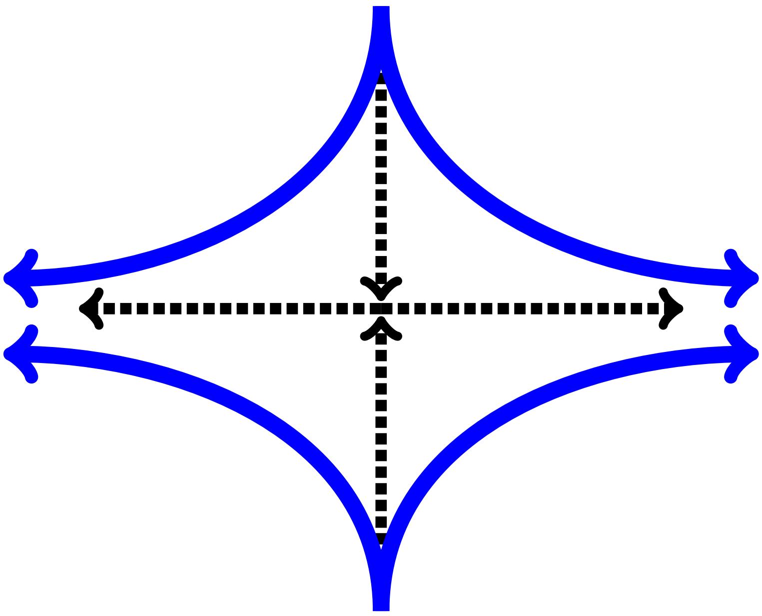
where κ is the magnetic diffusion coefficient inversely proportional to the fluid conductivity. We study here the kinematic regime when the back reaction of the magnetic field on the flow can be neglected.

A chaotic velocity field has several spatial scales at which its correlation properties differ significantly. The smallest one is the local length scale R such that for the distances less than R the velocity field is spatially smooth. Namely, in the vicinity $|\boldsymbol{r}-\boldsymbol{r}(t)| \ll R$ of a given Lagrangian trajectory $\boldsymbol{r}(t)$ the velocity can be approximated by a linear profile:

$$v_\mu(\boldsymbol{r},t) \approx V_\mu^{(0)}(t) + \sigma_{\mu\nu}(t) r_\nu, \quad V_\mu^{(0)}(t) = v_\mu(\boldsymbol{r}(t),t), \qquad (2)$$

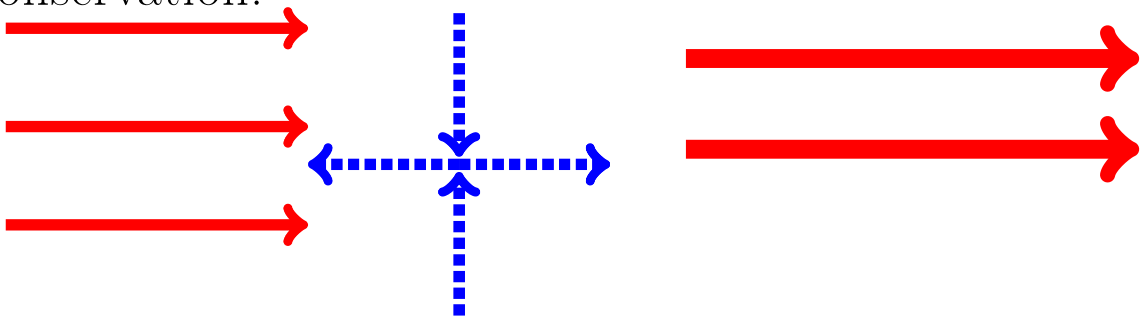
$$\sigma_{\mu\mu} = 0$$

Locally the flow is generally hyperbolic:



In the limit of infinite conductivity the magnetic field evolves in a neighborhood of a given Lagrangian particle locally. For a chaotic flow this evolution is a multiplicative matrix random process leading to exponential grows of the mean square amplitude of $\boldsymbol{B}$ independently on global properties of the flow and its dimensionality.

Enhancement of the field is a consequence of magnetic flux conservation:



If the resistance of the fluid is small but finite the dissipation governs large time behavior of the magnetic field distribution. It does not mean that the values of $\boldsymbol{B}^2, \boldsymbol{B}^4, \ldots$ averaged over the space cease growing. Rather, large-scale characteristics of the flow becomes important.

The magnetic diffusion is significant on the scales less then $r_d \sim \sqrt{\kappa/\lambda}$ where $\lambda \sim |\hat{\sigma}|$ is the characteristic Lyapunov exponent of divergence of close Lagrangian trajectories in the flow. We consider here the case when the ratio R/r_d: $R/r_d \gg 1$. This is natural relation between the parameters in astrophysical applications where R/r_d is proportional to square root $Pr_m^{1/2}$ of the magnetic Prandtl number Pr_m as well as in the polymer solutions where r_d originates from the molecular diffusion. We restrict ourselves to the case of small-scale initial magnetic field fluctuations when their correlation length l is small compared R: $r_d \lesssim l \ll R$. Then the magnetic diffusion determines the magnetic field evolution from the very beginning, or more precisely, for times t obeying the inequality $t > t_d = \lambda^{-1} \ln l/r_d \sim 1$. It is well-known that for three-dimensional flows the fluctuations of the field $\boldsymbol{B}$ continue to grow exponentially even in this dissipative regime having highly intermittent spatial and temporal statistics.

II. THE ORIGIN AND DURATION OF THE ENHANCEMENT STAGE IN THE MAGNETIC FIELD EVOLUTION: QUALITATIVE PICTURE.

It is easy to see from (1) that for the two-dimensional flow the transverse component B_3 of the magnetic field evolves independently on in-plane components $B_\alpha, \alpha = 1, 2$. In early papers of Zeldovich it was noted correctly that B_3 behaves like a decaying passive scalar field and goes to zero exponentially in an arbitrary point fixed in space. Then, as it is claimed in the cited works, the remaining two components should decay in the same manner. This conclusion is drawn from the consideration of the exact relation

$$\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = 0$$

together with the suppositions that all the terms in this relation have the same order of magnitude and the derivatives $\partial_1 B_1$ and $\partial_2 B_2$ have the same time dependence as the amplitudes B_1 and B_2. But the latter assumption is incorrect.

To show this let us consider the simple explicit example of the dissipative evolution of the initial data $\boldsymbol{B}^{(0)}(\boldsymbol{r})$ in the linear velocity field $v_1 = \lambda r_1$, $v_2 = -\lambda r_2$. In this case the evolution equation for the spatial Fourier components of the magnetic field

$$B_\alpha(\boldsymbol{r}, t) = \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{kr}} B^\alpha_{\boldsymbol{k}}(t) \tag{3}$$

becomes the first-order partial differential equation:

$$\partial_t B^\alpha_{\boldsymbol{k}} - \sigma_{\mu\nu} k_\mu \frac{\partial}{\partial k_\nu} B^\alpha_{\boldsymbol{k}} - B^\nu_{\boldsymbol{k}} \sigma_{\alpha\nu} + \kappa \boldsymbol{k}^2 B^\alpha_{\boldsymbol{k}} = 0. \tag{4}$$

where $\sigma_{\mu\nu}(t) = \mathrm{diag}(\lambda, -\lambda, 0)$. This equation can be solved easily by the characteristic method. For the sake of definiteness let us take the Gaussian profile for the initial field distribution:

$$\boldsymbol{B}(\boldsymbol{r}, t = 0) = \mathrm{curl}\boldsymbol{A}, \quad \boldsymbol{A} = \boldsymbol{a} \exp\left(-\frac{\boldsymbol{r}^2}{4l^2}\right), \boldsymbol{a} = (0, a, 0). \tag{5}$$

For this vector-potential the second field component is equal to zero: $B_2 = 0$. The time evolution of the component $B_1(\boldsymbol{r}, t)$ involves two asymptotical stages. The first one corresponds to the time interval $\lambda^{-1} \ll t \ll \lambda^{-1}\ln(l/r_d)$ and can be called the diffusion-free regime:

$$B_1 \approx \frac{ar_3e^{\lambda t}}{2l^2}\exp\left\{-\frac{r_3^2}{4l^2} - \frac{r_2^2e^{2\lambda t}}{4l^2} - \frac{r_1^2}{4l^2}e^{-2\lambda t}\right\}. \tag{6}$$

The initial blob is being stretched along the axis $(1, 0, 0)$. The transverse size of the blob decreases but the dissipative scale $r_d = \sqrt{2\kappa/\lambda}$ is not yet reached. As a result the field amplitude grows exponentially. The second stage correspond to the resistive regime and it takes place at $t \gg \lambda^{-1}\ln(l/r_d)$:

$$B_1 \approx \frac{ar_3}{lr_d\,(1+\kappa t/\,l^2)^{3/2}}\times \tag{7}$$

$$\times\exp\left\{-\frac{r_3^2}{4(l^2+\kappa t)} - \frac{r_2^2}{r_d^2} - \frac{r_1^2}{4l^2}e^{-2\lambda t}\right\}, \tag{8}$$

$$B_3 \approx -r_1\frac{ae^{-2\lambda t}}{lr_d\,(1+\kappa t/\,l^2)^{1/2}}\times \tag{9}$$

$$\times\exp\left\{-\frac{r_3^2}{4(l^2+\kappa t)} - \frac{r_2^2}{r_d^2} - \frac{r_1^2}{4l^2}e^{-2\lambda t}\right\}. \tag{10}$$

The field amplitude B_1 inside the blob stops to grow but it does not decay exponentially. Moreover, the field energy continues to grow: $\int d^3\boldsymbol{r}\boldsymbol{B}^2 \propto e^{\lambda t}$. It happens in spite of the fact that the component B_3 together with the two-dimensional divergence $\partial_1 B_1 + \partial_2 B_2 = \partial_1 B_1$ decay exponentially for any fixed $\boldsymbol{r}$. However, the maximal value of B_3 is reached in the point $\boldsymbol{r}(t) = (l\exp(2\lambda t), 0, 0)$ moving with time and this maximum decays more slowly than the value of B_3 at any fixed point. This «inflation» of the in-plane divergence of the field is the key property of the problem lost in the previous studies. Let us take now the initial condition in form of field blobs with centers $\boldsymbol{R}_{\mathbf{n}}$ randomly distributed over space:

$$\boldsymbol{B}(\boldsymbol{r}, 0) = \operatorname{curl}\left[\sum_{\boldsymbol{n}} \boldsymbol{a}_{\boldsymbol{n}} \exp\left(-\frac{(\boldsymbol{r} - \boldsymbol{R}_{\mathbf{n}})^2}{4l}\right)\right], \tag{11}$$

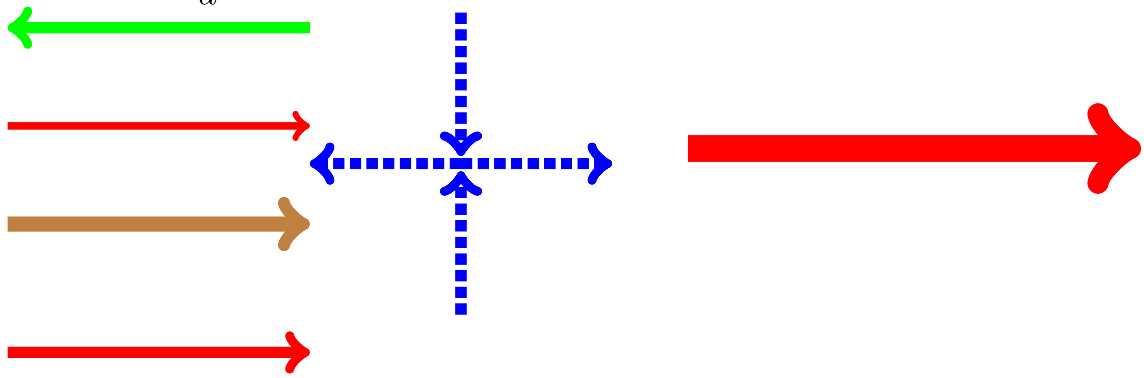
$$\boldsymbol{a}_{\boldsymbol{n}} = (0, a_{\boldsymbol{n}}, 0), \mathbf{n} = (n_1, n_2, n_3).$$

The amplitude $B_1(\boldsymbol{r}, t)$ at a given time moment $t \gg \lambda^{-1}\ln(L/r_d)$ can be represented as a sum of exponentially large number of addends:

$$B_1 \propto \sum_{-M}^{M} a_{(0,m,0)}, \qquad M \sim r_d e^{\lambda t}/L. \tag{12}$$

If $a_{(0,m,0)}$ are uncorrelated random numbers then we arrive to the estimation $B_1 \propto e^{\lambda t/2}$. For random matrix $\hat{\sigma}(t)$ in the expansion (2) the Lyapunov exponent λ is a fluctuating quantity. The moments of the field $\boldsymbol{B}$ can be evaluated with the same estimation $|\boldsymbol{B}| \propto e^{\lambda t/2}$ averaged over the statistics of λ: $\langle \boldsymbol{B}^{2n}(t)\rangle \propto \langle \exp(n\lambda t)\rangle$. The resulting statistics of the magnetic field is intermittent because of the inequality $\langle \exp(n\lambda t)\rangle \gg \exp(n\langle\lambda\rangle t)$ for $\lambda t \gg 1$.

The exponential growth of the magnetic field fluctuations continues in the resistive regime due to aggregation of the initial blobs along the contracting direction $(0, 1, 0)$ of the flow. The magnetic diffusion homogenizes the field on the scales $\sim r_d$.



One can check easily that for a pure two-dimensional magnetic field (e.g., for $\boldsymbol{a} = (0, 0, a)$ in the example (5)) the moments of $\boldsymbol{B}$ decay exponentially at $t \gg \lambda^{-1} \ln l/r_d$. The velocity field providing the exponential increase $|\boldsymbol{B}| \propto e^{\lambda t/2}$ grows at spatial infinity and cannot be used for a global of any physical flow. On the other hand, flows on compact two-dimensional manifolds do not produce unlimited growth of the magnetic field. To understand how the compactness of the space affects the long time behavior of the field note that the enhancement of $\langle \boldsymbol{B}^2 \rangle$ in the example considered above is the result of coalescence of the field filaments with *uncorrelated* amplitudes. But for a finite size of the system or for a finite correlation length R of the velocity field this property may cease to hold for large enough λt. Indeed, in the course of evolution the magnetic lines of force form narrow strip-like clusters with the widths $\sim r_d$ and exponentially growing lengths. The field $\boldsymbol{B}$ is flattening out in the stretching direction and becomes strongly correlated along such strips. The correlation length R of the velocity field turns out to be the characteristic curvature radius of these strips. As a result the magnetic field becomes correlated along random curves in the plane. Hence, several parts of the same strip fall within the contracting domain of the flow. The anticorrelation arising in this way with certain probability in the field distribution can either modify the growth of the field moments or stop it at all. This phenomenon is illustrated schematically in Fig.1. One can note some similarity of the phenomenon with weak localization.

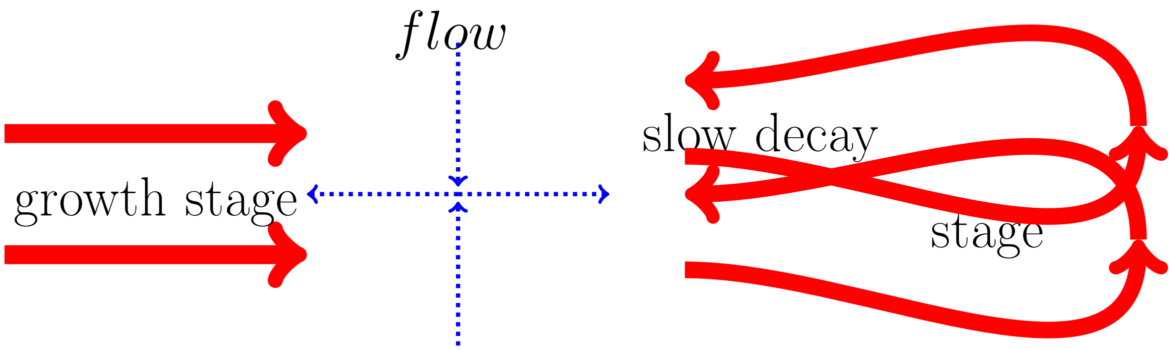


Fig.1. This figure illustrates the anticorrelation mechanism terminating the dynamo in two-dimensional flow: the hyperbolic flow on the growth stage is stretching the magnetic field blobs into filament (thick arrows on the left part) and contracting these filaments with arbitrary correlated field values (completely correlated thick arrows on the left part of the figure) in transverse direction forming one filament with growing field strongly correlated along the filament. When the lengths of the resulting filaments reach the flow scale R (the radius of curvature of the arrows in the right part of the picture) their shapes become twisted what finally lead to anticorrelation in the spatial field distribution.

III. THE KRAICHNAN-KAZANTSEV MODEL IN TWO DIMENSIONS.

The analytic description of the phenomenon is developed here for the Kraichnan-Kazantsev model where the velocity $\boldsymbol{v}(\boldsymbol{r},t)$ statistics is supposed to be Gaussian with zero mean value and the pair correlator of the form:

$$\langle v_\mu(\boldsymbol{r},t) v_\nu(\boldsymbol{r'},t')\rangle = \delta(t-t')\mathcal{C}_{\mu\nu}(\boldsymbol{r}-\boldsymbol{r'}). \tag{13}$$

We consider the initial magnetic field to be randomly distributed over the space. The evolution equation for the correlation tensor $F_{\alpha\beta}(\boldsymbol{r},t) = \langle B_\alpha(\boldsymbol{r'},t) B_\beta(\boldsymbol{r'}+\boldsymbol{r},t)\rangle$ follows directly from the equation of motion (1):

$$\partial_t F_{\alpha\beta} = \left[\mathcal{C}_{\mu\nu}(0) - \mathcal{C}_{\mu\nu}(\boldsymbol{r})\right] \partial_\mu \partial_\nu F_{\alpha\beta} +$$

$$+\partial_\mu \mathcal{C}_{\nu\beta}(\boldsymbol{r}) \partial_\nu F_{\alpha\mu} + \partial_\mu \mathcal{C}_{\nu\alpha}(\boldsymbol{r}) \partial_\nu F_{\mu\beta} - \tag{14}$$

$$-F_{\mu\nu}\partial_\mu \partial_\nu \mathcal{C}_{\alpha\beta}(\boldsymbol{r}) + 2\kappa \triangle F_{\alpha\beta}. \tag{15}$$

The magnetic field has all three components depending on the third coordinate r_3 as well. On the other hand, the tensor $\mathcal{C}_{\mu\nu}$ does not depend on r_3 and the corresponding components of $\mathcal{C}_{\mu\nu}$ are equal to zero: $\mathcal{C}_{\mu 3} = \mathcal{C}_{3\nu} = 0$. In this case there is a closed evolution equation for the in-plane tensor $F_{\alpha\beta}, \alpha, \beta = 1, 2$ and just this tensor is considered below. This reduced correlation tensor has non-zero divergency: $\partial_\alpha F_{\alpha\beta} \neq 0$. The coordinate r_3 is a parameter in the two-dimensional problem and it is not pointed out in the sequel explicitly.

The tensor function $\mathcal{C}_{\mu\nu}(\boldsymbol{r})$ depends smoothly on $\boldsymbol{r}$ and goes to zero at $r \gg R$. This one-scale behavior can be realized in two-dimensional turbulent flow. Nevertheless, it must be emphasized that for a class of physical situations like small-scale dynamo and passive scalar decay it is precisely the smoothness domain $r \leq R$ which governs long time asymptotics.

We consider the statistics of the velocity field to be isotropic:

$$\mathcal{C}_{\mu\nu}(\boldsymbol{r}) = \delta_{\mu\nu}\mathcal{C}_1(r) + \frac{r_\mu r_\nu}{r^2}\mathcal{C}_2(r). \tag{16}$$

The incompressibility of the flow leads to the following relation between $\mathcal{C}_1(r)$ and $\mathcal{C}_2(r)$:

$$\mathcal{C}_1'(r) = -\mathcal{C}_2'(r) - \frac{1}{r}\mathcal{C}_2(r), \tag{17}$$

$$\mathcal{C}_1(r) = V_0 R - \mathcal{C}_2(r) - \int\limits_0^r \frac{dr'}{r'}\mathcal{C}_2(r'), \quad \mathcal{C}_2(0) = 0.$$

The value V_0 has the sense of the typical advection velocity and is defined by the condition $\mathcal{C}_1(r \to \infty) = \mathcal{C}_2(r \to \infty) = 0$. The statistical isotropy of the magnetic field leads to the decomposition of the tensor $F_{\alpha\beta}(\boldsymbol{r}, t)$ similar to (16):

$$F_{\alpha\beta}(\boldsymbol{r}, t) = \delta_{\alpha\beta}\mathcal{S}(r, t) + \frac{r_\alpha r_\beta}{r^2}\mathcal{Y}(r, t). \tag{18}$$

One can check that the evolution of the function

$$\Phi(r, t) = (1 + r\partial_r)\mathcal{Y}(r, t) + r\partial_r\mathcal{S}(r, t), \quad \partial_\alpha F_{\alpha\beta} = \frac{r_\alpha}{r^2}\Phi(r, t), \tag{19}$$

decouples:

$$\partial_t\Phi = [\mathcal{C}_1(0) - \mathcal{C}_1(r) - \mathcal{C}_2(r)]\left(\partial_r^2\Phi - \frac{1}{r}\partial_r\Phi\right) + r_d^2\left(\partial_r^2\Phi - r^{-1}\partial_r\Phi\right). \tag{20}$$

The enhancement of the magnetic field is determined by the dynamics at the scales $r \ll R$ where one can use the expansion:

$$\langle v_\mu(\boldsymbol{r}, t)v_\nu(\boldsymbol{0}, t')\rangle \approx \left[V_0 R\delta_{\mu\nu} - \lambda\left(3r^2\delta_{\mu\nu} - 2r_\mu r_\nu\right)\right]\delta(t - t'). \tag{21}$$

The Lyapunov exponent λ can be expressed as $\lambda = V_0/R$. If (21) works the equations for the functions $\Phi(r, t)$ and $Y(r, t)$ have the form:

$$\lambda^{-1}\partial_t\Phi = \hat{\mathcal{L}}_\Phi\Phi, \quad \hat{\mathcal{L}}_\Phi = r^2\partial_r^2 - r\partial_r + r_d^2\left(\partial_r^2 - r^{-1}\partial_r\right), \tag{22}$$

$$\lambda^{-1}\partial_t\mathcal{Y} = \hat{\mathcal{L}}_\mathcal{Y}\mathcal{Y} - 8\Phi, \quad \hat{\mathcal{L}}_\mathcal{Y} = r^2\partial_r^2 + 7r\partial_r + 8 + r_d^2\left(\partial_r^2 + r^{-1}\partial_r - 4r^{-2}\right). \tag{23}$$

The dissipative scale $r_d = \sqrt{2\kappa/\lambda}$ is considered to be smallest in the problem. At largest distances $r \gg R$ the correlation tensor obey the diffusion equation:

$$\partial_t F_{\alpha\beta} = D\triangle F_{\alpha\beta}, \quad D = \lambda R^2. \tag{24}$$

To solve the evolution equations we use the Laplace transform:

$$\mathcal{Y}_p(r) = \lambda\int\limits_0^\infty dt\, e^{-p\lambda t}\mathcal{Y}(r, t), \tag{25}$$

The function $\Phi_p(r)$ defined analogously is equal to the convolution of the initial data $\Phi^{(0)}(r) = \Phi(r, t = 0)$ with the Green function:

$$\Phi_p(r) = \int_0^\infty dr'\, G_\Phi(p|r, r')\Phi^{(0)}(r'), \quad G_\Phi(p|r, r') = \left\langle r \left| (p - \hat{\mathcal{L}}_\Phi)^{-1} \right| r' \right\rangle . \tag{26}$$

The Eq.(23) for the function $\mathcal{Y}(r, t)$ at $r \ll R$ has the source $-8\Phi(r, t)$. Thus the Laplace transform $\mathcal{Y}_p(r)$ is expressed both in terms of the initial condition $\mathcal{Y}^{(0)}(r) = \mathcal{Y}(r, t = 0)$ and the source $\Phi_p(r)$:

$$\mathcal{Y}_p(r) \approx \int_0^R dr'\, G_{\mathcal{Y}}(p|r, r') \left(\mathcal{Y}^{(0)}(r') - \Phi_p(r') \right), \quad G_{\mathcal{Y}}(p|r, r') = \left\langle r \left| (p - \hat{\mathcal{L}}_{\mathcal{Y}})^{-1} \right| r' \right\rangle . \tag{27}$$

We present derivation of $\Phi(r, t)$ and $\mathcal{Y}(r, t)$; the function $\mathcal{S}(r, t)$ can be restored easily.

We are interesting in the behavior of $\Phi(r, t)$ and $\mathcal{Y}(r)$ at $r_d \ll r \ll R$ and large times. The initial distributions $\Phi^{(0)}(r)$ and $\mathcal{Y}(r)^{(0)}$ are supposed to be smooth functions localized on the scale l with the same order of magnitude:$\Phi^{(0)}(0) \sim \mathcal{Y}(0)^{(0)} \sim f_0$. To avoid multiplication of intermediate asymptotics we consider here the case $l \sim r_d$. The role of the magnetic diffusion is crucial in formation of long-time asymptotics. But looking for $\Phi(r, t)$ and $\mathcal{Y}(r)$ at $r_d \ll r \ll R$ and $\lambda t \gg 1$ we see that the this role is reduced to stabilization of their temporal behavior at $r \lesssim r_d$. Technically this means that we can omit the diffusive part in the operators (22) and (23) cutting off the dr'-integration in (26) and (27) by the distance r_d. The Green functions $G_{\Phi,\mathcal{Y}}(p|r, r')$ at $r_d \ll r \ll R$ are constructed of zero modes of the operators $p - \hat{\mathcal{L}}_{\Phi,\mathcal{Y}}$. Such zero modes are linear combinations of power functions with coefficients defined by matching zero modes at larger distances $r \geq R$. It is not attainable to compute these coefficients exactly but to determine the long-time behavior of $F_{\alpha\beta}(\boldsymbol{r}, t)$ only rightest singularities of the coefficients in the complex p-plane are needed. Unboundedness of this right domain together with the diffusive type of the evolution produce the singularities at the point $p = 0$. The computational details can be found in the Appendix. Here we present the results for the components of the magnetic field correlation tensor in various space and time intervals.

There are two asymptotical regimes in the evolution of the function $\Phi(r, t)$. During the first one corresponding to $2\lambda t < \ln R/r_d$ the shape of $\Phi(r, t)$ may be described as an exponentially blowing hull:

$$\Phi(r, t) \propto \frac{f_0}{\sqrt{\lambda t}} \frac{r}{r_d} \exp\left(-\lambda t - \frac{1}{4\lambda t} \ln^2 \frac{r}{r_d} \right) \tag{28}$$

One can see that $\Phi(r, t)$ is concentrated in a narrow neighborhood of the running point $r_m(t) = r_d \exp(2\lambda t)$; the value $\Phi_m = \Phi(r_m(t), t)$ decreases slowly: $\Phi_m \sim f_0(\lambda t)^{-1/2}$. When $r_m(t)$ reaches the velocity scale R the behavior of $\Phi(r, t)$ changes:

$$\Phi(r, t) \propto f_0 \frac{r^2}{R^2} \frac{1}{(\lambda t)^2}, \quad \lambda t \gg \ln R/r \gg 1. \tag{29}$$

Turning to the function $\mathcal{Y}(r, t)$ note first that the the initial data $\mathcal{Y}^{(0)}(r)$ contribution decays monotonically as $\sim \exp(-\lambda t)$ at the first stage of the evolution $1 \ll \lambda t \ll \ln(R/r)$ and it is ignored below. It is worth emphasizing that this is the only contribution in the case of two-dimensional magnetic field. In our case the long-time evolution of $\mathcal{Y}(r, t)$ is determined by the source term in (23). The kernel in (27) grows with r' on the real semi-axis $p < 3$. In its turn the source $\Phi_p(r)$ grows with r and as a result of these two factors $\mathcal{Y}(r, t)$ increases exponentially at intermediate times:

$$\mathcal{Y}(r, t) \propto f_0 \frac{l}{r} \exp(3\lambda t), \quad \frac{1}{4} \ln \frac{r}{r_d} \ll \lambda t \leq \frac{1}{4} \ln \frac{R^2}{lr}. \tag{30}$$

In the next time inerval the grows of $\mathcal{Y}(r, t)$ continues but it becomes R - dependent:

$$\mathcal{Y}(r, t) \propto \frac{f_0}{\sqrt{\lambda t}} \frac{R^4}{r^3 l} \exp\left(-\lambda t - \frac{1}{4\lambda t} \ln^2 \frac{R^2}{rl} \right), \quad \frac{1}{4} \ln \frac{R^2}{lr} \leq \lambda t \leq \frac{1}{2} \ln \frac{R^2}{lr}, \tag{31}$$

The maximal value reached by $\mathcal{Y}(r, t)$ is parametrically large: $\mathcal{Y}_{max}(r) \sim R^2/r^2$. The behavior of $\mathcal{Y}(r, t)$ at $\lambda t \gg \ln(R/r)$ is governed by the singularity of $\mathcal{Y}_p(r)$ at $p = 0$:

$$\mathcal{Y}(r, t) \propto f_0 \frac{R^2}{r^2} \frac{1}{(\lambda t)^2}, \quad \lambda t \gg \ln \frac{R^2}{lr}. \tag{32}$$

The complete correlation tensor at large times and $r \ll R$ is restored noting that Φ can be neglected in the relation (19):

$$F_{\alpha\beta}(r,t) \propto \frac{f_0}{(\lambda t)^2}\left(-\delta_{\alpha\beta} + 2\frac{r_\alpha r_\beta}{r^2}\right)\frac{R^2}{r^2}, \quad r_d \ll r \ll R, \tag{33}$$

$$F_{\alpha\beta}(r,t) \propto \frac{f_0}{(\lambda t)^2}\frac{R^2}{r_d^2}\delta_{\alpha\beta}, \quad r \lesssim r_d.$$

These estimations are the main result.

IV. CONCLUSION.

We see that in two-dimensional chaotic flow the fluctuations of the magnetic field increase their amplitude in R/r_d times. The statistics of the field in the course of the exponential growth is highly intermittent similar to the three-dimensional case: the field energy is concentrated in strip-like domains with widths of the order of r_d.

Let us consider now the four-point correlation tensor. It is easy to see that there is the decomposition:

$$\langle B_\alpha(\boldsymbol{r}_1,t)B_\beta(\boldsymbol{r}_2,t)B_\gamma(\boldsymbol{r}_3,t)B_\mu(\boldsymbol{r}_4,t)\rangle = \hat{T}\left(\boldsymbol{R}_{12},\boldsymbol{R}_{34}\right) + \tag{34}$$

$$\hat{T}\left(\boldsymbol{R}_{13},\boldsymbol{R}_{24}\right) + \hat{T}\left(\boldsymbol{R}_{14},\boldsymbol{R}_{23}\right),$$

where $\boldsymbol{R}_{jl} = \boldsymbol{r}_j - \boldsymbol{r}_l$, $j,l = 1,\dots,4$ and for small distances $R_{1,2} \ll r_d$ we have:

$$\left[\hat{T}\left(\boldsymbol{R}_1,\boldsymbol{R}_2\right)\right]_{\alpha\beta,\mu\lambda} \propto \left(\delta_{\alpha\beta}\delta_{\mu\lambda} + \delta_{\alpha\mu}\delta_{\beta\lambda} + \delta_{\alpha\lambda}\delta_{\beta\mu}\right)\left(\frac{cf_0}{lr_d}\right)^2 \exp\left(4\mathcal{G}t\right). \quad r \ll r_d, \tag{35}$$

If $R_1 \sim R_2 \sim R \gg r_d$ the correlation tensor $\hat{T}\left(\boldsymbol{R}_1,\boldsymbol{R}_2\right)$ demonstrates strong angular dependence; for finite Θ_{12} where $\Theta_{12} = \varphi_{\boldsymbol{R}_1} - \varphi_{\boldsymbol{R}_2}$ is the angle between $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ it is exponentially small:

$$\hat{T}\left(\boldsymbol{R}_1,\boldsymbol{R}_2\right) \sim \exp\left(-\frac{R^2}{r_d^2}\Theta_{12}^2\right). \tag{36}$$

In the collinear limit $\boldsymbol{R}_{1,2} = \boldsymbol{n}R_{1,2}$ one can get the simple expression:

$$\hat{T}\left(\boldsymbol{R}_1, \boldsymbol{R}_2\right) \sim n_\alpha n_\beta n_\gamma n_\mu \frac{f_0^2 l^2}{r_d\sqrt{R_1^2 + R_2^2}} \exp\left(4\mathcal{G}t\right). \qquad (37)$$

After $t \sim \lambda^{-1}\ln(R/r_d)$ the growth gives way to the slow decrease.

APPENDIX A: GREEN FUNCTIONS AND INVERSE LAPLACE TRANSFORM

Here we present explicit expressions for the Green functions $G_{\Phi,\mathcal{Y}}(p|r,r')$ and outline briefly the inverse Laplace transform leading to the Eqs.(28)-(33).

For $r \geq r', r' \ll R$ the Green function $G_\Phi(p|r,r')$ has the form:

$$G_\Phi(p|r,r') = \varphi_p(r)\frac{(r')^{-2+\sqrt{p+1}}}{2\sqrt{p+1}}, \qquad \text{(A1)}$$

where $\varphi_p(r)$ at $r \ll R$ is a linear superposition of power functions:

$$\varphi_p(r) = r^{1-\sqrt{p+1}} + b_p R^{-2\sqrt{p+1}} r^{1+\sqrt{p+1}}, \quad r \ll R, \qquad \text{(A2)}$$

with the coefficient b_p to be determined by matching the Laplace transform of a homogeneous solution of Eq. (20) in the the domain $r \geq R$. For $r \gg R$ and $Re\, p > 0$ the function $\varphi_p(r)$ is a decaying solution of the equation:

$$\left(p/R^2 - \partial_r^2 + r^{-1}\partial_r\right)\varphi_p(r) = 0, \qquad \text{(A3)}$$

and has the form

$$\varphi_p(r) \approx \mathcal{B}_p R^{-\sqrt{p+1}} r K_1\left(\frac{r}{R}\sqrt{p}\right), \quad r \gg R. \qquad \text{(A4)}$$

The solution in the domain $r \sim R$ defines 2×2-matrix $\hat{g}$ which is the transition matrix between the asymptotics (A2) and (A4). It fixes the coefficients $\mathcal{B}_p$ and b_p:

$$\left(b_p + 1, b_p + 1 + \sqrt{p+1}(b_p - 1)\right)\hat{g} = \mathcal{B}_p\left(K_1(\sqrt{p}), -\sqrt{p}K_0(\sqrt{p})\right). \qquad \text{(A5)}$$

For regular functions $\mathcal{C}_{1,2}(r)$ the elements of $\hat{g}$ have no singular points in the p-plane. Thence the singularities of b_p and $\mathcal{B}_p$ determining large time behavior emerge from the right hand side of (A5) and are placed at $p = 0$. The leading terms in expansion of b_p near $p = 0$ are

$$p \to 0, \quad b_p \to b_0 + b_1 p\ln p, \quad b_{0,1} \sim 1. \qquad \text{(A6)}$$

If $r \gg R$ and $p \to 0$ the Laplace transform $\varphi_p(r)$ is proportional to the modified Bessel function:

$$p \to 0, \quad \varphi_p \propto \frac{r}{R}pK_1\left(\frac{r}{R}\sqrt{p}\right). \qquad \text{(A7)}$$

The main contribution to the integral (26) is given by $r' \sim r_d$. In inverting the Laplace transform the line of p-integration can be deformed to upper and lower edges of the cut along the negative real semi-axes. For the times $2\lambda t < \ln R/r_d$ the behavior of $\Phi(r,t)$ is determined by the contribution of vicinity of the point $p = -1-(\lambda t)^{-2}(\ln r/r_d)^2$ leading to (28).

The Green function $G_\mathcal{Y}(p|r,r')$ can be found in the same way; for for $r'>r$ and $r,r'\ll R$ it has the form:

$$G_\mathcal{Y}(p|r,r') = r^{-3+\sqrt{p+1}}\,\frac{(r')^{2-\sqrt{p+1}}}{2\sqrt{p+1}}\left[1+a_p\left(\frac{r'}{R}\right)^{2\sqrt{p+1}}\right]. \tag{A8}$$

The coefficient a_p is defined like b_p by matching to the solution in the domain $r\geq R$ and it has the logarithmic branching point at $p=0$:

$$p\to 0,\quad a_p\to a_0+a_1p^2\ln p,\quad a_{0,1}\sim 1. \tag{A9}$$

As it was mentioned above the contribution $\sim\mathcal{Y}^{(0)}(r)$ can be ignored. The source contribution has the form:

$$\begin{aligned}
&\mathcal{Y}_p(r)=\mathcal{Y}_p^A(r)+\mathcal{Y}_p^B(r),\\
&\mathcal{Y}_p^A(r)\sim\frac{f_0 r^{1-\sqrt{p+1}}}{(p+1)(2-\sqrt{p+1})}\left(\frac{R}{r}\right)^{4-\sqrt{p+1}}\left[1-\left(\frac{r}{R}\right)^{4-\sqrt{p+1}}\right],\\
&\mathcal{Y}_p^B(r)\sim\frac{f_0 r^{1-\sqrt{p+1}}}{(p+1)(2+\sqrt{p+1})}\left(a_p+b_p+\frac{2a_pb_p}{2+\sqrt{p+1}}\right)\left(\frac{R}{r}\right)^{4-\sqrt{p+1}}.
\end{aligned} \tag{A10}$$

In the formal limit $R\to\infty$ the term $\mathcal{Y}_p^A(r)$ has the singularity at $p=3$. For any finite R there is no true singularity at $p=3$ and the exponential growth is an intermediate asymptotics (30). It is instructive to compare the situation with three-dimensional model [18]. In this case the closed equation for the trace $\mathcal{F}=F_{\alpha\alpha}$ can be derived:

$$\lambda^{-1}\partial_t\mathcal{F}=\hat{\mathcal{L}}_\mathcal{F}\mathcal{F},\quad \hat{\mathcal{L}}_\mathcal{F}=r^2\partial_r^2+6r\partial_r+10,\quad r_d\ll r\ll R. \tag{A11}$$

It can be checked easily that the Laplace transform $\mathcal{F}_p(r)$ has the true singularity at $p=15/4$:

$$\mathcal{F}_p(r)\propto f_0\left(\frac{L}{r}\right)^{5/2}\frac{1}{\sqrt{p-15/4}}\left[1-\left(\frac{r}{R}\right)^{\sqrt{p-15/4}}\right].$$

The cancellation of the divergence does not lead to elimination of the branching point at $p=15/4$.

The long-time behavior (33) results from the singularity of the coefficient b_p at $p=0$. It is worth noting that the expression (33) corresponds to the non-negative transverse spatial Fourier transform $F_{\alpha\beta}(\boldsymbol{k},t)$ of the correlation tensor:

$$F_{\alpha\beta}(\boldsymbol{k},t)\propto\frac{f_0R^2}{(\lambda t)^2}\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{\boldsymbol{k}^2}\right),\quad k_\alpha F_{\alpha\beta}(\boldsymbol{k},t)=0.$$

---

[1] H.K. Moffatt, Magnetic field generation in electrically conducting fluids, Cambridge Univ. Press, 1978.
[2] E.N. Parker, Cosmic magnetic fields, their origin and activity, Clarendon Press, Oxford, 1979.
[3] Ya.B.Zeldovich, A.A.Ruzmaikin, D.D.Sokolov, The Almighty Chance, World Scientific, 1990.
[4] S. Childress and A. Gilbert, Stretch, Twist, Fold: The Fast Dynamo, Springer-Verlag, Berlin, 1995.
[5] L.D. Landau, E.M. Lifshitz, Course of theoretical physics, Electrodynamics of Continuous Media, Pergamon, 1984.
[6] S.I. Vainshtein, Ya.B. Zel'dovich, Sov.Phys. Usp., **15**, 159 (1972).
[7] A.A.Schekochihin, S.C.Cowley, S.F.Taylor, J.L.Maron, J.C.McWilliams, The Astrophysical Journal, **612**, 276 (2004).
[8] A. Groisman, V. Steinberg, Nature **405,** 53 (2000).
[9] T.Burghelea, E.Segre, V.Steinberg, Phys. of Fluids, **19**, 053104 (2007).
[10] U.Frisch, Turbulence. The legacy of A.N.Kolmogorov, Cambridge Univ. Press, 1995.
[11] H. Xia, M. Shats, and G. Falkovich, Phys. Fluids **21**, 125101 (2009).
[12] Q. Chen, S. Chen, G. Eyink, and D. Holm, J. Fluid Mech. **542**, 139 (2005).
[13] G. Boffetta and R. E. Ecke, Ann. Rev. Fluid Mech. **44**, 427 (2012).
[14] E. Deusebio, G. Boffetta, E. Lindborg, S. Musacchio, Phys. Rev., E **90**, 023005 (2014).
[15] G. Falkovich and V. Lebedev, Phys. Rev. E **49**, R1800 (1994); **50**, 3883 (1994).
[16] H. Furstenberg, Trans. Amer. Math. Soc. **108**, 377 (1963).
[17] A. Crisanti, G. Paladin, A. Vulpiani, *Products of Random Matrices* (Springer-Verlag, Berlin 1993).
[18] A.P. Kazantsev, Sov. Phys JETP **26**, 1031 (1968).
[19] V.G.Novikov, A.A.Ruzmaikin, D.D.Sokolov, Sov.Phys. JETP, **58**, 527 (1983).

[20] M. Chertkov, G. Falkovich, I. Kolokolov, M. Vergassola, Phys. Rev. Lett. **83**, 4065 (1999).
[21] G. Falkovich, K. Gawedzki, and M. Vergassola Rev. Mod. Phys. **73**, 913 (2001).
[22] Zel'dovich Ya. B., Sov. Phys. JETP **4**, 460 (1957).
[23] Ya. B. Zel'dovich, A.A. Ruzmaikin, JETP, **51**, 493, (1980).
[24] D. D. Sokoloff, R. A. Stepanov, P. G. Frick, Phys. Usp. **57**, 292-311 (2014).
[25] V.R. Kogan, I.V. Kolokolov and V.V. Lebedev, J. Phys. A: Math. Theor. **43**, 182001,(2010)
[26] A.A.Schekochihin, S.A.Boldyrev and R.M.Kulsrud, The Astrophysical Journal, **567**, 828 (2002).
[27] V.Oseledets, Geophysical and Astrophysical Fluid Dynamics, **73**, 133 (1993).
[28] V.I.Arnold, B.A.Khesin, Topological Methods in Hydrodynamics, Springer-Verlag New York, Inc., 1998.
[29] R. Kraichnan and S. Nagarajan, Phys. Fluids **10**, 859 (1967).
[30] A. I. Larkin, D. E. Khmel'nitskii, Sov. Phys. Uspekhi, **25**, 185-187 (1982).
[31] A.A.Schekochihin, S.C.Cowley, S.F.Taylor, G.W.Hammett, J.L.Maron, and J.C.McWilliams, Phys. Rev. Lett. **92**, 084504 (2004).
[32] P.H.Haynes, J.Vanneste, Phys. of Fluids, **17**, 097103 (2005).
[33] D.Vincenzi, J. of Stat. Phys., **106,** 1073 (2002).