Dynamics of Tensor and SYK Models

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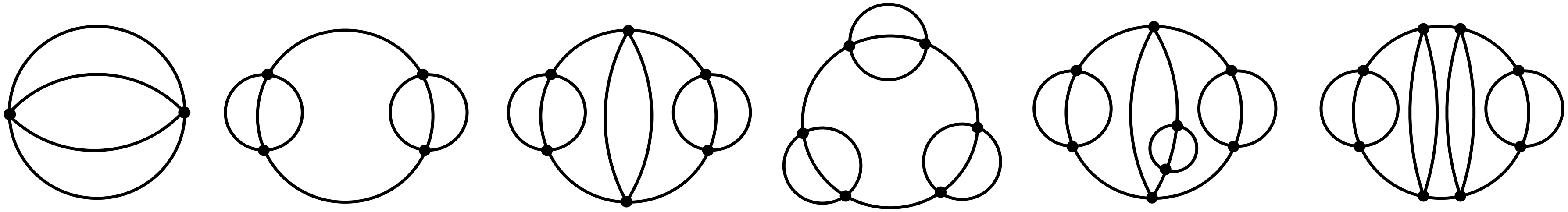
Talk at the Khalatnikov Conference
Chernogolovka, October 18, 2019

- IK, Grigory Tarnopolsky, arXiv: 1611.08915
- IK, Fedor Popov, Grigory Tarnopolsky,
  "TASI Lectures on Large N Tensor Models,"
  arXiv: 1808.09434
- Jaewon Kim, IK, Grigory Tarnopolsky, Wenli Zhao, arXiv:1902.02287, Physical Review X9 (2019) 021043

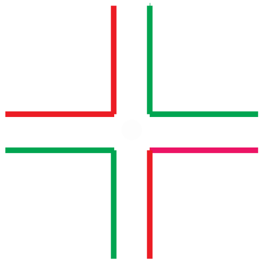
Three Large N Limits

- O(N) Vector: solvable because the "cactus" diagrams can be summed. 
- Matrix ('t Hooft) Limit: planar diagrams. Solvable only in special cases.
- Tensor of rank three and higher. When interactions are specially chosen, dominated by the "melonic" diagrams. Bonzom, Gurau, Riello, Rivasseau; Carrozza, Tanasa; Witten; IK, Tarnopolsky

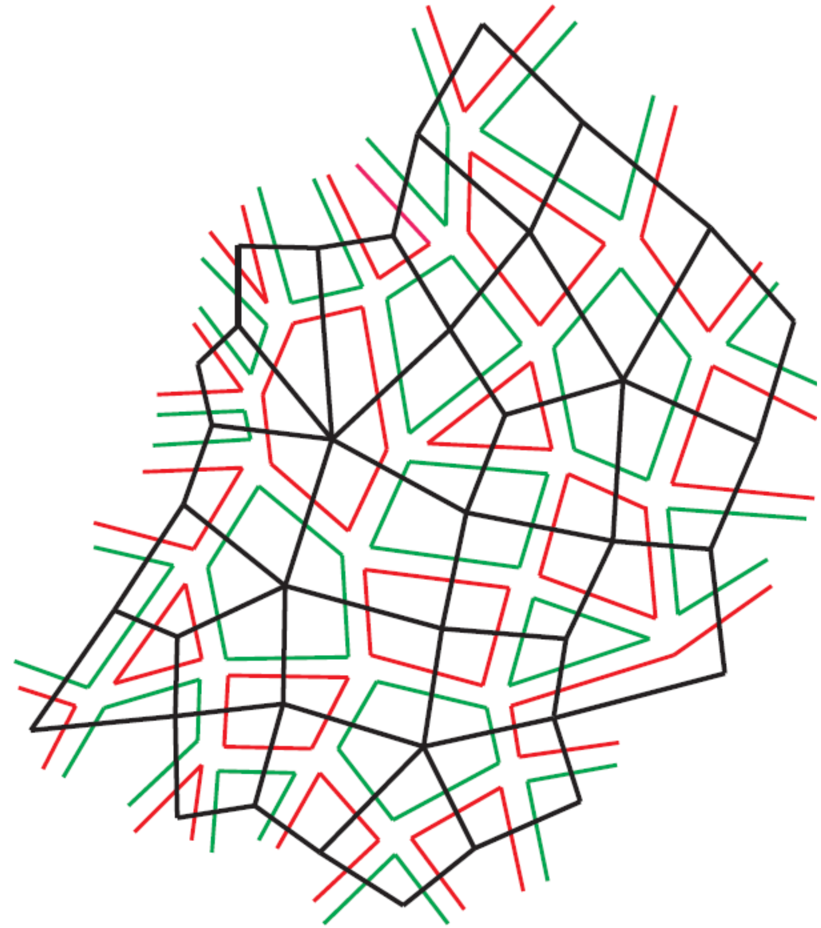


O(N) x O(N) Matrix Model

- Theory of real matrices ϕ^{ab} with distinguishable indices, i.e. in the bi-fundamental representation of $O(N)_a x O(N)_b$ symmetry.
- The interaction is at least quartic: $g \, \mathrm{tr} \, \phi\phi^T\phi\phi^T$
- Propagators are represented by colored double lines, and the interaction vertex is
- In d=0 or 1 special limits describe two-dimensional quantum gravity.



- In the large N limit where gN is held fixed we find planar Feynman graphs, and each index loop may be red or green.
- The dual graphs shown in black may be thought of as random surfaces tiled with squares whose vertices have alternating colors (red, green, red, green).



From Bi- to Tri-Fundamentals

- For a 3-tensor with distinguishable indices the propagator has index structure

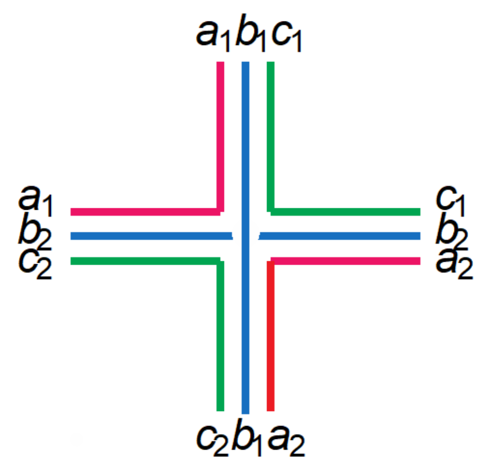
$$\langle \phi^{abc} \phi^{a'b'c'} \rangle = \delta^{aa'} \delta^{bb'} \delta^{cc'}$$

- It may be represented graphically by 3 colored wires 

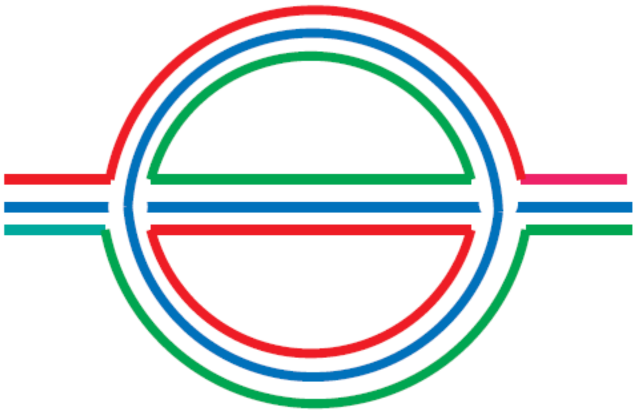
- Tetrahedral interaction with $O(N)_a \times O(N)_b \times O(N)_c$ symmetry

Carrozza, Tanasa; IK, Tarnopolsky

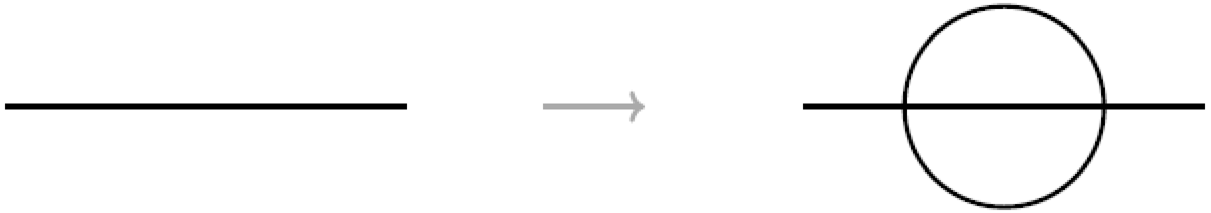
$$\frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1}$$



- Leading correction to the propagator has 3 index loops

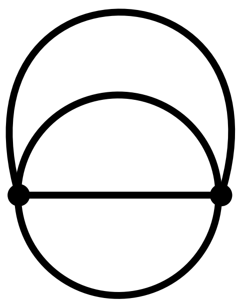


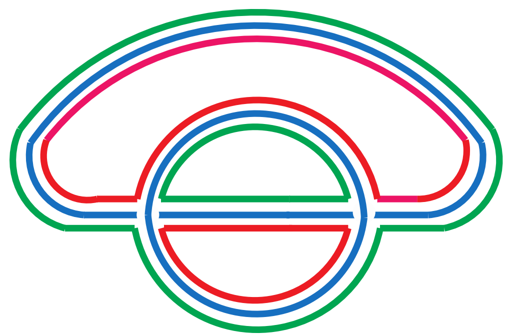
- Requiring that this “melon” insertion is of order 1 means that $\lambda = gN^{3/2}$ must be held fixed in the large N limit.
- Melonic graphs obtained by iterating



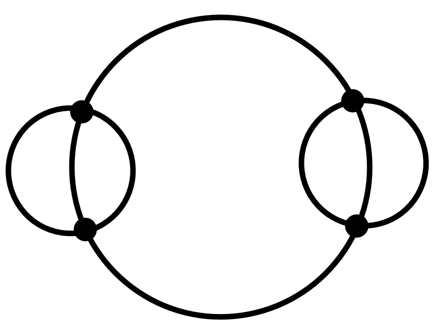
Cables and Wires

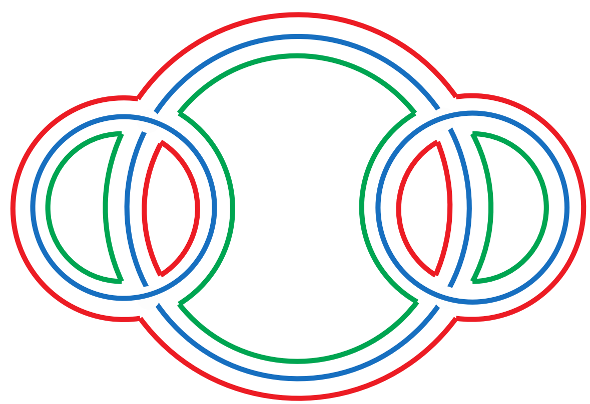
- The Feynman graphs of the quartic field theory may be resolved in terms of the colored wires (triple lines)





$$g^2 N^6 \sim N^3 \lambda^2$$

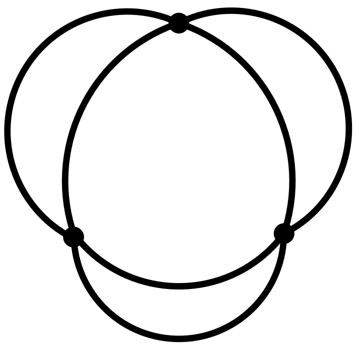


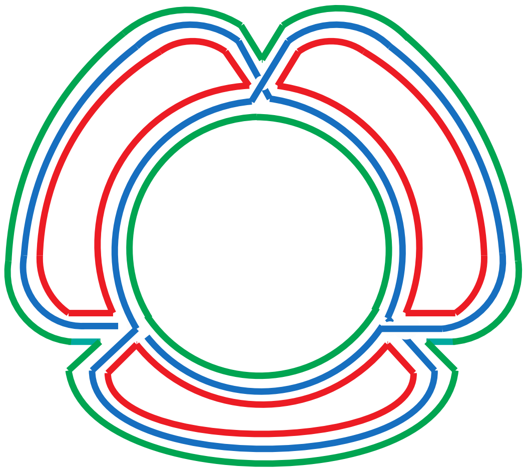


$$g^4 N^9 \sim N^3 \lambda^4$$

Non-Melonic Graphs

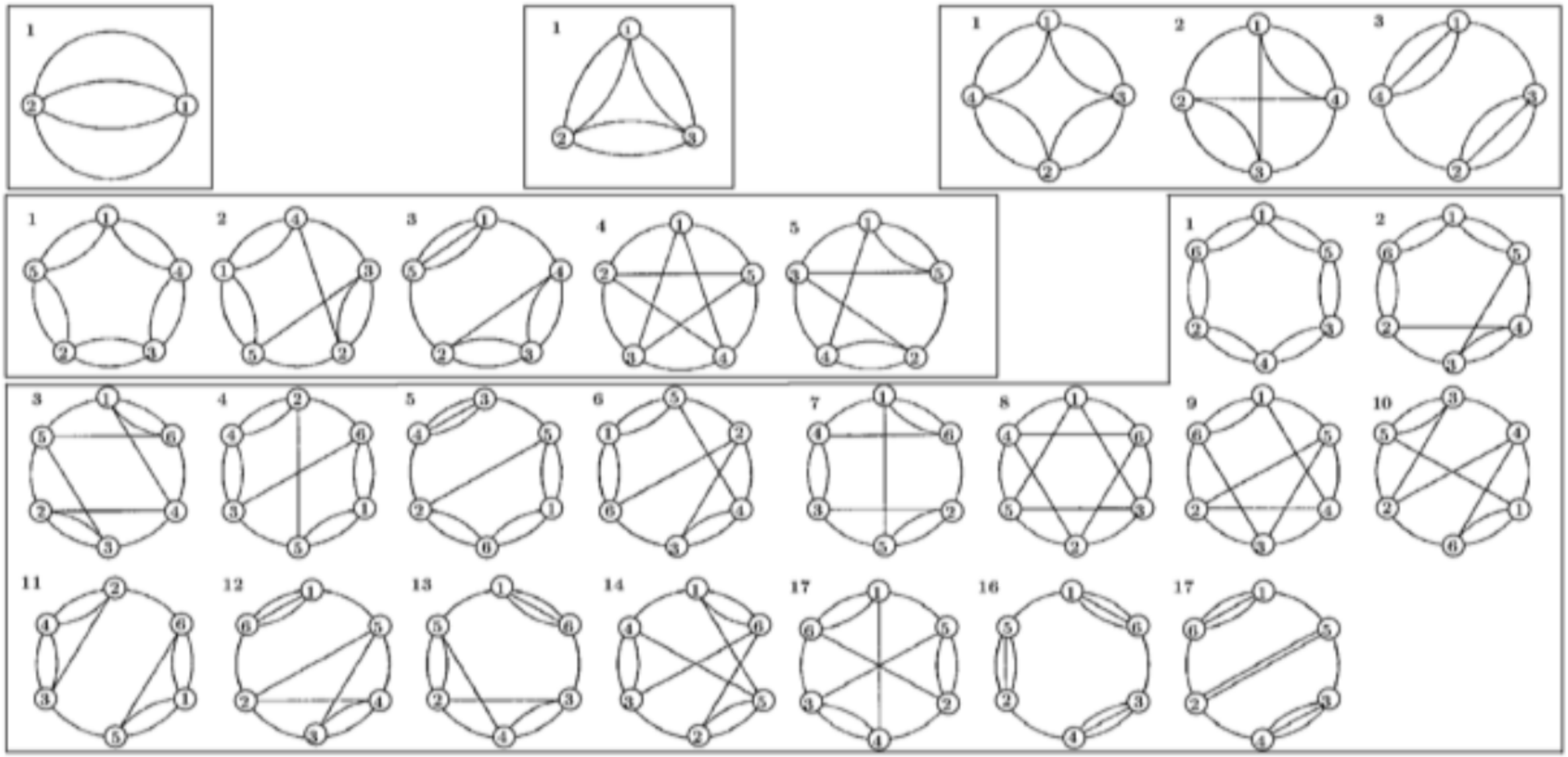
- Most Feynman graphs in the quartic field theory are not melonic are therefore subdominant in the new large N limit, e.g.





- Scales as $g^3 N^6 \sim N^3 \lambda^3 N^{-3/2}$
- None of the graphs with an odd number of vertices are melonic.

- Here is the list of snail-free vacuum graphs up to 6 vertices Kleinert, Schulte-Frohlinde



- Only 4 out of these 27 graphs are melonic.
- The number of melonic graphs with p vertices grows as C^p Bonzom, Gurau, Riello, Rivasseau

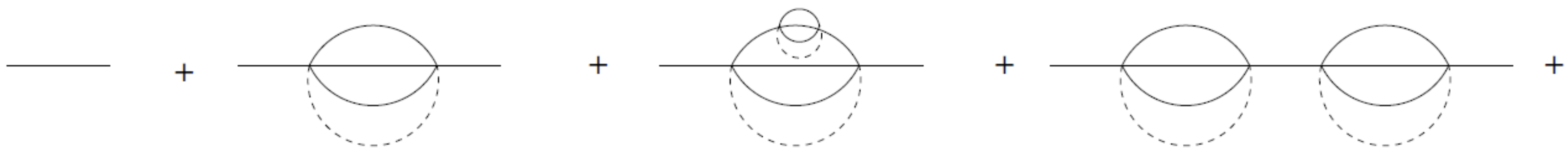
The Sachdev-Ye-Kitaev Model

- Quantum mechanics of a large number N_{SYK} of anti-commuting variables with action

$$I = \int \mathrm{d}t \left(\frac{\mathrm{i}}{2} \sum_i \psi_i \frac{\mathrm{d}}{\mathrm{d}t} \psi_i - \mathrm{i}^{q/2} j_{i_1 i_2 \dots i_q} \psi_{i_1} \psi_{i_2} \dots \psi_{i_q} \right)$$

- Random couplings j have a Gaussian distribution with zero mean.
- The model flows to strong coupling and becomes nearly conformal. Sachdev, Ye; Georges, Parcollet, Sachdev; Kitaev; Polchinski, Rosenhaus; Maldacena, Stanford; Jevicki, Suzuki, Yoon; Kitaev, Suh

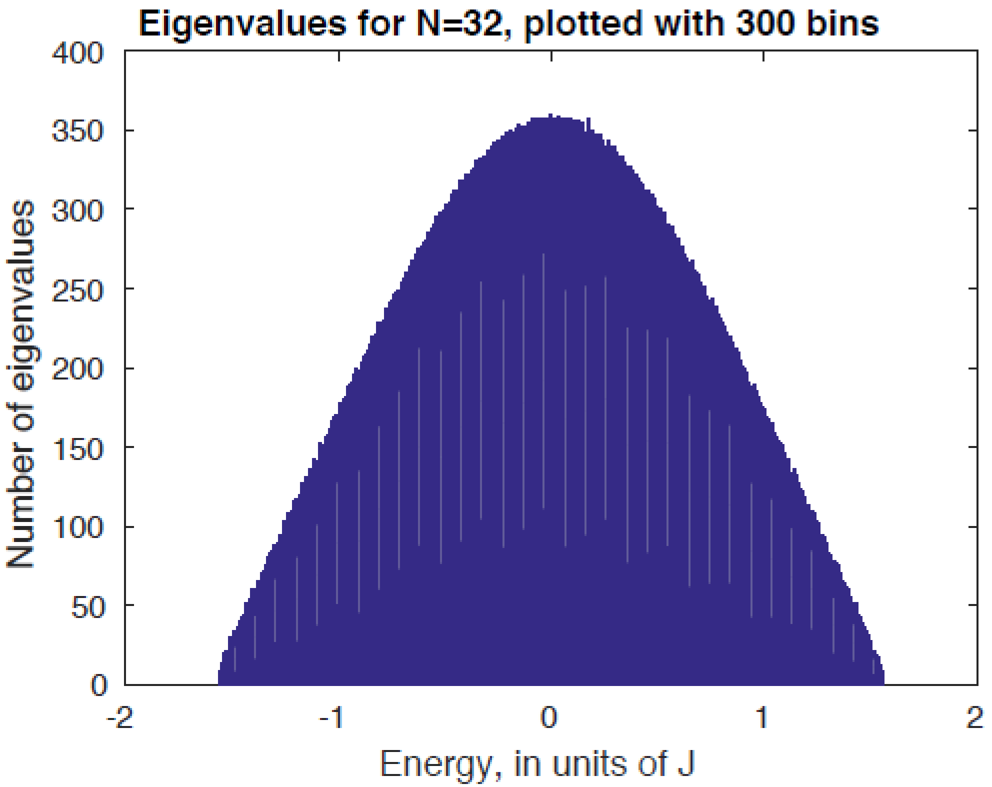
- The simplest dynamical case is q=4.
- Exactly solvable in the large N_{SYK} limit because only the melonic Feynman diagrams contribute



- Solid lines are fermion propagators, while dashed lines mean disorder average.
- The exact solution shows resemblance with physics of certain two-dimensional black holes.

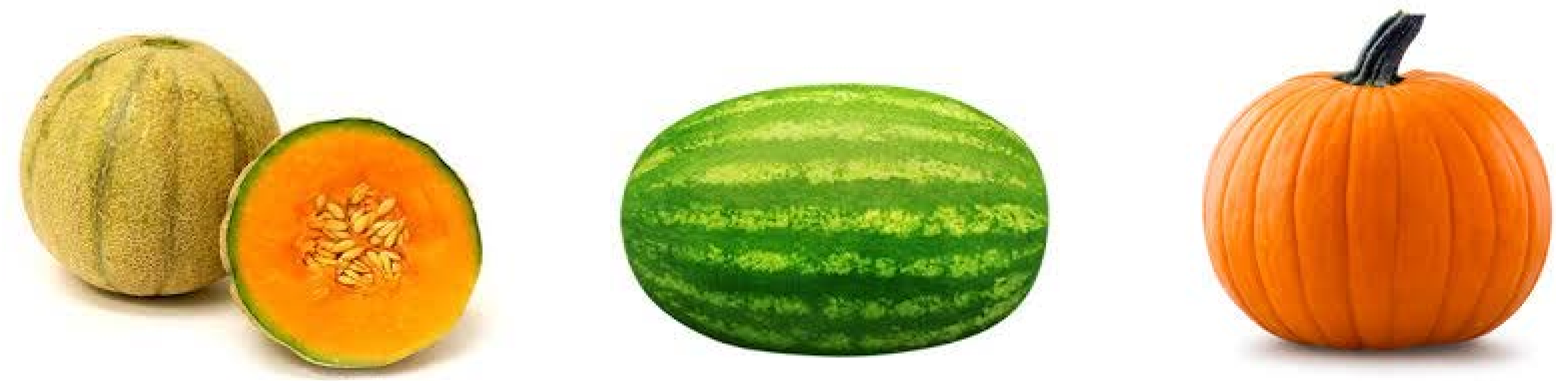
  Kitaev; Almheiri, Polchinski; Sachdev; Maldacena, Stanford, Yang; Engelsoy, Mertens, Verlinde; Jensen; Kitaev, Suh; …

- Spectrum for a single realization of N_{SYK}=32 model with q=4. Maldacena, Stanford
- No exact degeneracies, but the gaps are exponentially small. Large low T entropy.



Majorana Tensor QM

- E. Witten, "An SYK-Like Model Without Disorder," arXiv: 1610.09758.
- Appeared on the evening of Halloween: October 31, 2016.



- It is sometimes tempting to change the term "melonic diagrams" to "pumpkinlike diagrams."

The Gurau-Witten Model

- This model is called "colored" in the random tensor literature because the anti-commuting 3-tensor fields ψ_A^{abc} carry a label A=0,1,2,3.

$$S_{\text{Gurau}-\text{Witten}} = \int dt \Big(\frac{i}{2} \psi_A^{abc} \partial_t \psi_A^{abc} + g \psi_0^{abc} \psi_1^{ade} \psi_2^{fbe} \psi_3^{fdc} \Big)$$

- Perhaps more natural to call it "flavored."
- The model has $O(N)^6$ symmetry with each tensor in a tri-fundamental under a different subset of the six symmetry groups.
- Contains $4N^3$ Majorana fermions.

The O(N)^3 Model

- A pruned version: there are N^3 Majorana fermions IK, Tarnopolsky

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N^4$$

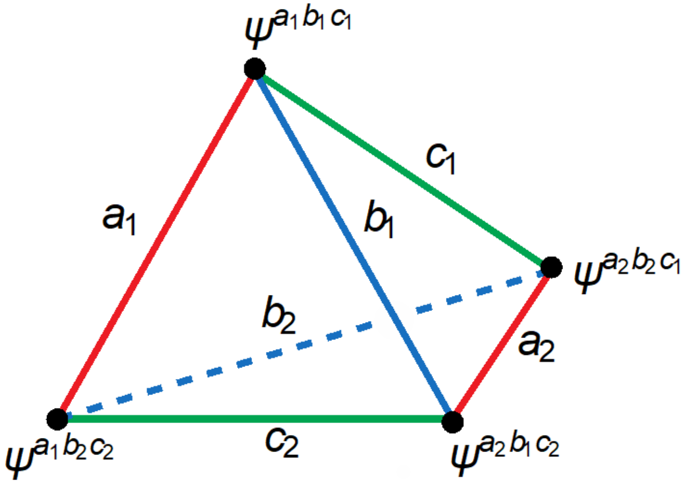
- Has $O(N)_a$x$O(N)_b$x$O(N)_c$ symmetry under

$$\psi^{abc} \to M_1^{aa'} M_2^{bb'} M_3^{cc'} \psi^{a'b'c'}, \qquad M_1, M_2, M_3 \in O(N)$$

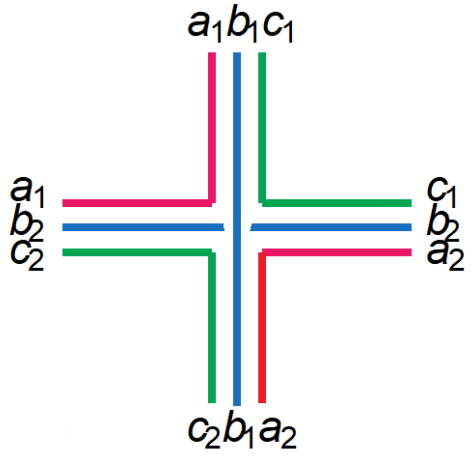
- The SO(N) symmetry charges are

$$Q_1^{aa'} = \frac{i}{2}[\psi^{abc}, \psi^{a'bc}], \qquad Q_2^{bb'} = \frac{i}{2}[\psi^{abc}, \psi^{ab'c}], \qquad Q_3^{cc'} = \frac{i}{2}[\psi^{abc}, \psi^{abc'}]$$

- The 3-tensors may be associated with indistinguishable vertices of a tetrahedron.

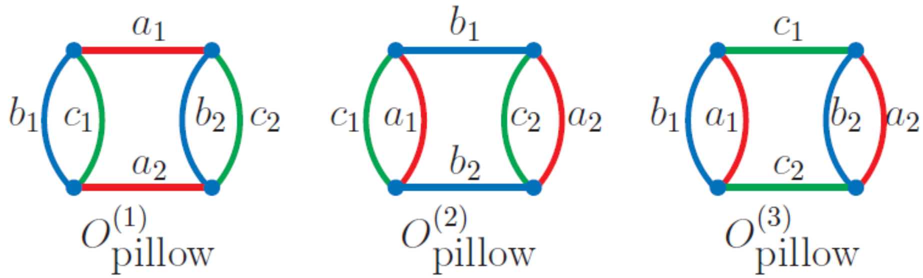


- This is equivalent to



- The triple-line Feynman graphs are produced using the propagator

- The tetrahedral term is the unique dynamical quartic interaction with O(N)^3 symmetry.
- The other possible terms are quadratic Casimirs of the three SO(N) groups.



$$O^{(1)}_{\text{pillow}} = \sum_{a_1<a_2} Q_1^{a_1a_2} Q_1^{a_1a_2} , \qquad O^{(2)}_{\text{pillow}} = \sum_{b_1<b_2} Q_2^{b_1b_2} Q_2^{b_1b_2} , \qquad O^{(3)}_{\text{pillow}} = \sum_{c_1<c_2} Q_3^{c_1c_2} Q_3^{c_1c_2}$$

- In the model where SO(N)^3 is gauged, they vanish.

$O(N)^3$ vs. SYK Model

- Using composite indices $I_k = (a_k b_k c_k)$

$$H = \frac{1}{4!} J_{I_1 I_2 I_3 I_4} \psi^{I_1} \psi^{I_2} \psi^{I_3} \psi^{I_4}$$

The couplings take values $0, \pm 1$

$$J_{I_1 I_2 I_3 I_4} = \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \delta_{c_1 c_4} \delta_{c_2 c_3} - \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_2 b_3} \delta_{b_1 b_4} \delta_{c_2 c_4} \delta_{c_1 c_3} + 22 \text{ terms}$$

- The number of distinct terms is

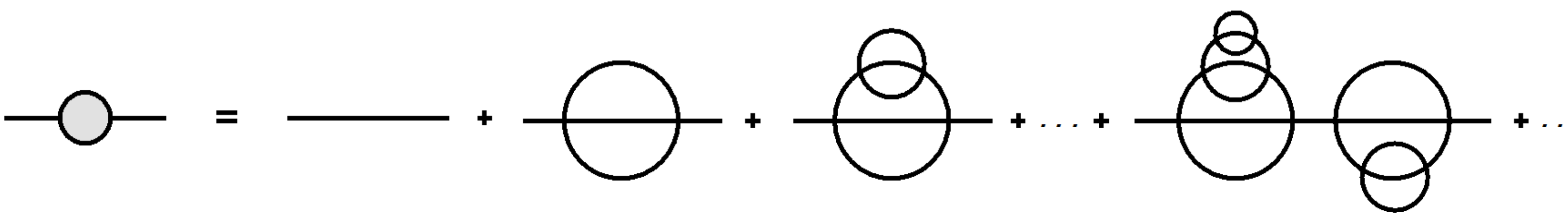
$$\frac{1}{4!} \sum_{\{I_k\}} J^2_{I_1 I_2 I_3 I_4} = \frac{1}{4} N^3 (N-1)^2 (N+2)$$

- Much smaller than in SYK model with $N_{\text{SYK}} = N^3$

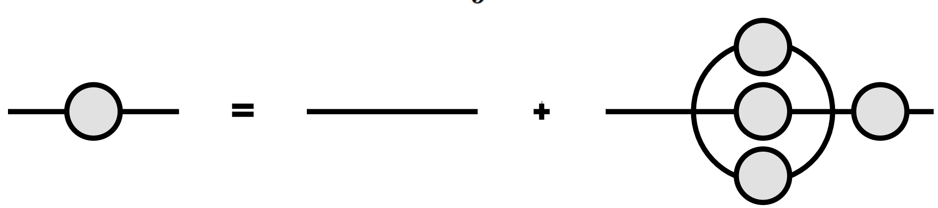
$$\frac{1}{24} N^3 (N^3 - 1)(N^3 - 2)(N^3 - 3)$$

Schwinger-Dyson Equations

- Some are the same as in the SYK model Kitaev; Polchinski, Rosenhaus; Maldacena, Stanford; Jevicki, Suzuki, Yoon; Kitaev, Suh



$$G(t_1 - t_2) = G_0(t_1 - t_2) + g^2 N^3 \int dt dt' G_0(t_1 - t) G(t - t')^3 G(t' - t_2)$$

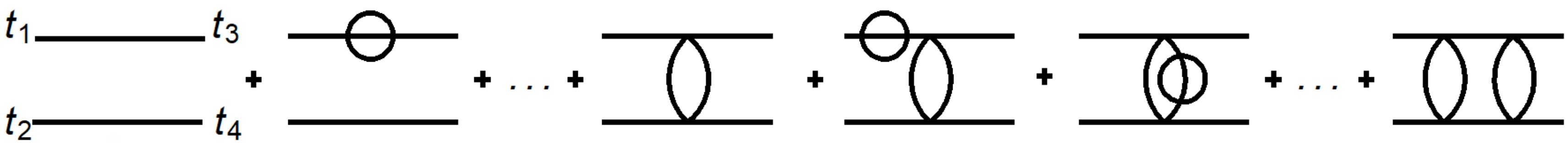


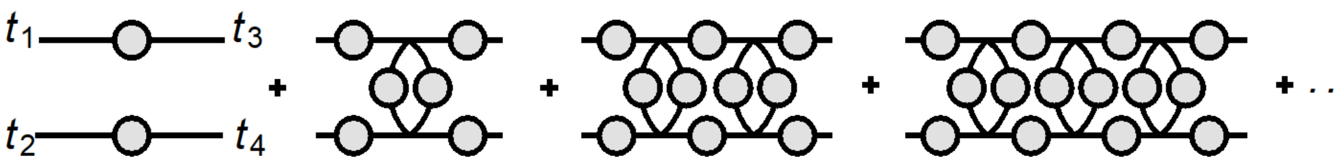
- Neglecting the left-hand side in IR we find

$$G(t_1 - t_2) = -\left(\frac{1}{4\pi g^2 N^3}\right)^{1/4} \frac{\mathrm{sgn}(t_1 - t_2)}{|t_1 - t_2|^{1/2}}$$

- Four point function

$$\langle \psi^{a_1 b_1 c_1}(t_1)\psi^{a_1 b_1 c_1}(t_2)\psi^{a_2 b_2 c_2}(t_3)\psi^{a_2 b_2 c_2}(t_4)\rangle = N^6 G(t_{12})G(t_{34}) + \Gamma(t_1,\ldots,t_4)$$





- If we denote by Γ_n the ladder with n rungs

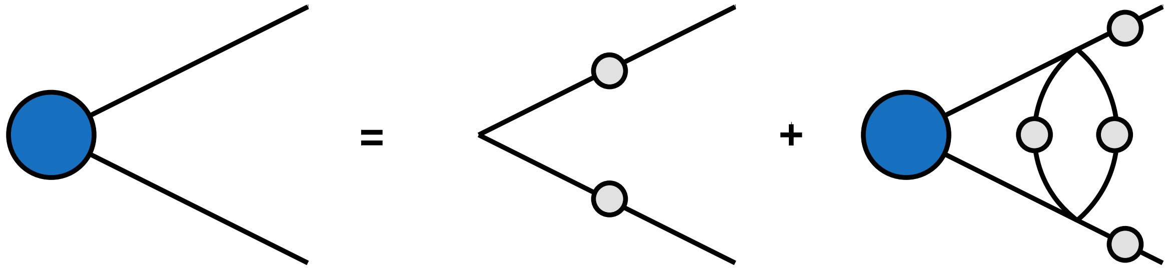
$$\Gamma = \sum_n \Gamma_n$$

$$\Gamma_{n+1}(t_1,\ldots,t_4) = \int dt dt' K(t_1,t_2;t,t')\Gamma_n(t,t',t_3,t_4)$$

$$K(t_1,t_2;t_3,t_4) = -3g^2 N^3 G(t_{13})G(t_{24})G(t_{34})^2$$

Spectrum of two-particle operators

- S-D equation for the three-point function Gross, Rosenhaus



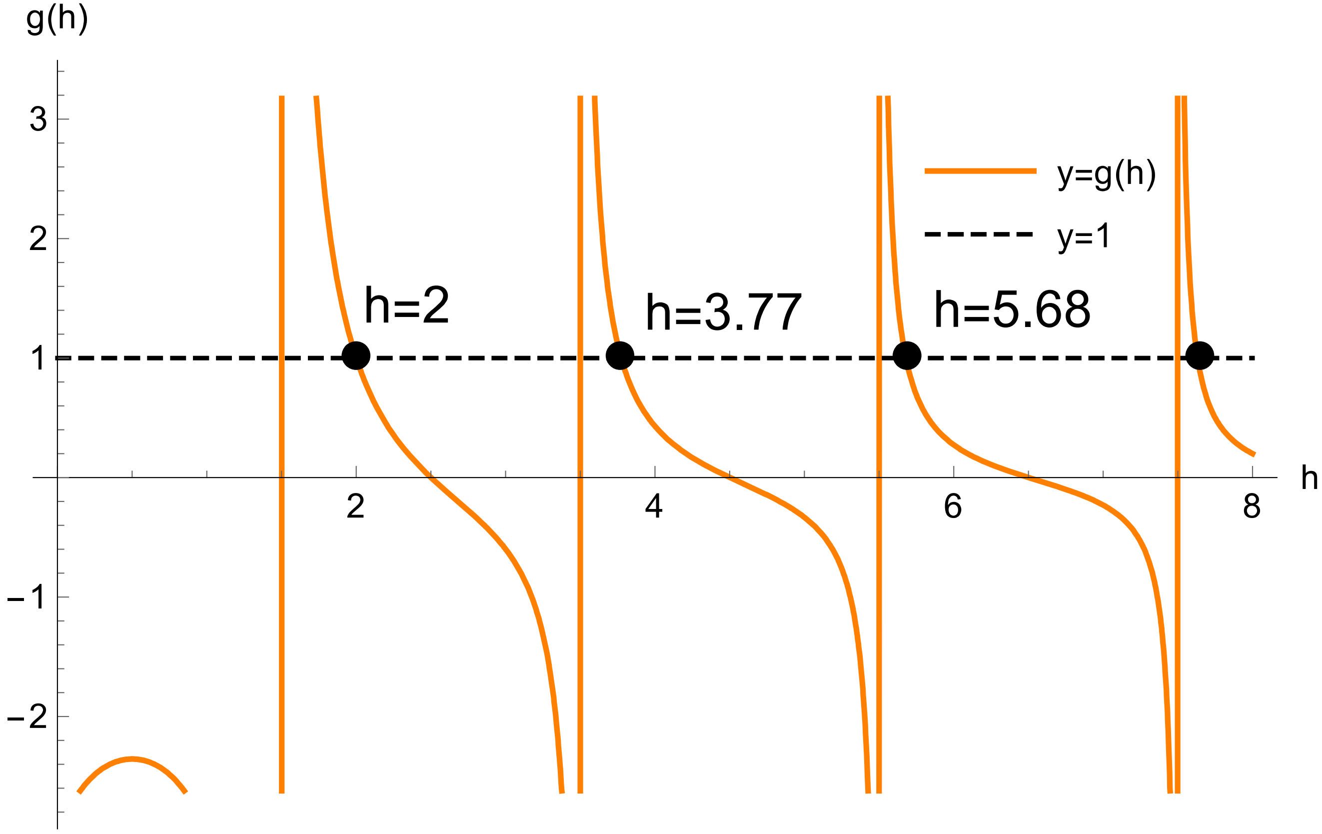
$$v(t_0, t_1, t_2) = g(h) \int dt_3 dt_4 K(t_1, t_2; t_3, t_4) v(t_0, t_3, t_4)$$

$$v(t_0, t_1, t_2) = \langle O_2^n(t_0) \psi^{abc}(t_1) \psi^{abc}(t_2) \rangle = \frac{\mathrm{sgn}(t_1 - t_2)}{|t_0 - t_1|^h |t_0 - t_2|^h |t_1 - t_2|^{1/2-h}}$$

- Scaling dimensions of operators $O_2^n = \psi^{abc} (D_t^n \psi)^{abc}$

$$g(h) = -\frac{3}{2} \frac{\tan(\frac{\pi}{2}(h - \frac{1}{2}))}{h - 1/2} = 1$$

- The first solution is h=2; dual to JT gravity.



- The higher scaling dimensions are

$h \approx 3.77,\ 5.68,\ 7.63,\ 9.60$ approaching $h_n \to n + \frac{1}{2}$

Tetrahedral Bosonic QFT

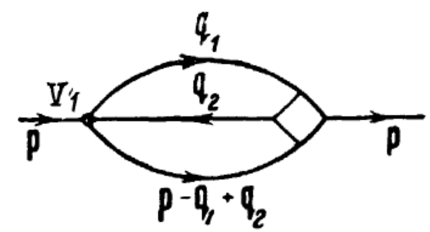
- Action with a potential that is not positive definite IK, Tarnopolsky; Giombi, IK, Tarnopolsky

$$S = \int d^d x \Big(\frac{1}{2} \partial_\mu \phi^{abc} \partial^\mu \phi^{abc} + \frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1} \Big)$$

- Schwinger-Dyson equation for 2pt function
  Patashinsky, Pokrovsky (1963)

$$G^{-1}(p) = -\lambda^2 \int \frac{d^d k d^d q}{(2\pi)^{2d}} G(q) G(k) G(p+q+k)$$

- Has solution



$$G(p) = \lambda^{-1/2} \Big(\frac{(4\pi)^d d \Gamma(\frac{3d}{4})}{4 \Gamma(1-\frac{d}{4})} \Big)^{1/4} \frac{1}{(p^2)^{\frac{d}{4}}}$$

Spectrum of two-particle spin zero operators

- Schwinger-Dyson equation

$$\int d^d x_3 d^d x_4 K(x_1, x_2; x_3, x_4) v_h(x_3, x_4) = g(h) v_h(x_1, x_2)$$

$$K(x_1, x_2; x_3, x_4) = 3\lambda^2 G(x_{13}) G(x_{24}) G(x_{34})^2$$

$$v_h(x_1, x_2) = \frac{1}{[(x_1 - x_2)^2]^{\frac{1}{2}(\frac{d}{2} - h)}}$$

$$g_{\text{bos}}(h) = -\frac{3\Gamma\left(\frac{3d}{4}\right)\Gamma\left(\frac{d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{h}{2} - \frac{d}{4}\right)}{\Gamma\left(-\frac{d}{4}\right)\Gamma\left(\frac{3d}{4} - \frac{h}{2}\right)\Gamma\left(\frac{d}{4} + \frac{h}{2}\right)}$$

- In d<4 the first solution is complex $\frac{d}{2} + i\alpha(d)$

Complex Fixed Point in 4-ε Dimensions

- The tetrahedron

$$O_t(x) = \phi^{a_1b_1c_1}\phi^{a_1b_2c_2}\phi^{a_2b_1c_2}\phi^{a_2b_2c_1}$$

mixes at finite N with the pillow and double-sum operators

$$O_p(x) = \frac{1}{3}\left(\phi^{a_1b_1c_1}\phi^{a_1b_1c_2}\phi^{a_2b_2c_2}\phi^{a_2b_2c_1} + \phi^{a_1b_1c_1}\phi^{a_2b_1c_1}\phi^{a_2b_2c_2}\phi^{a_1b_2c_2} + \phi^{a_1b_1c_1}\phi^{a_1b_2c_1}\phi^{a_2b_1c_2}\phi^{a_2b_2c_2}\right),$$

$$O_{ds}(x) = \phi^{a_1b_1c_1}\phi^{a_1b_1c_1}\phi^{a_2b_2c_2}\phi^{a_2b_2c_2}$$

- The renormalizable action is

$$S = \int d^dx\left(\frac{1}{2}\partial_\mu\phi^{abc}\partial^\mu\phi^{abc} + \frac{1}{4}\left(g_1O_t(x) + g_2O_p(x) + g_3O_{ds}(x)\right)\right)$$

- The large N scaling is

$$g_1 = \frac{(4\pi)^2 \tilde{g}_1}{N^{3/2}}, \quad g_2 = \frac{(4\pi)^2 \tilde{g}_2}{N^2}, \quad g_3 = \frac{(4\pi)^2 \tilde{g}_3}{N^3}$$

- The 2-loop beta functions and fixed points:

$$\tilde{\beta}_t = -\epsilon \tilde{g}_1 + 2\tilde{g}_1^3 ,$$

$$\tilde{\beta}_p = -\epsilon \tilde{g}_2 + \left(6\tilde{g}_1^2 + \frac{2}{3}\tilde{g}_2^2\right) - 2\tilde{g}_1^2 \tilde{g}_2 ,$$

$$\tilde{\beta}_{ds} = -\epsilon \tilde{g}_3 + \left(\frac{4}{3}\tilde{g}_2^2 + 4\tilde{g}_2\tilde{g}_3 + 2\tilde{g}_3^2\right) - 2\tilde{g}_1^2(4\tilde{g}_2 + 5\tilde{g}_3)$$

$$\tilde{g}_1^* = (\epsilon/2)^{1/2}, \quad \tilde{g}_2^* = \pm 3i(\epsilon/2)^{1/2}, \quad \tilde{g}_3^* = \mp i(3 \pm \sqrt{3})(\epsilon/2)^{1/2}$$

- The scaling dimension of $\phi^{abc}\phi^{abc}$ is

$$\Delta_O = d - 2 + 2(\tilde{g}_2^* + \tilde{g}_3^*) = 2 \pm i\sqrt{6\epsilon} + \mathcal{O}(\epsilon)$$

A Richer Set of Models

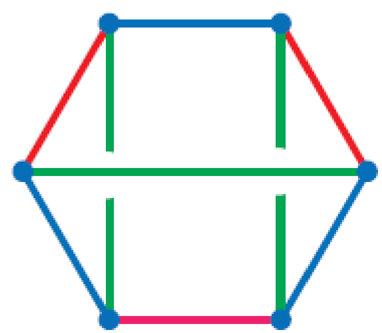
- The tetrahedral interaction is the simplest possibility of obtaining a solvable large N tensor model.
- There are many others!
- For the interaction of order 2n the maximal tensor rank is 2n-1. When it is lower, the theory may be called "subchromatic." Prakash, Sinha
- It is helpful to choose the dominant interaction to be Maximally Single Trace (MST). Ferrari, Rivasseau, Valette; IK, Pallegar, Popov

Prismatic Bosonic QFT

- Large N limit dominated by the positive sextic "prism" interaction Giombi, IK, Popov, Prakash, Tarnopolsky

$$S=\int d^dx\left(\frac{1}{2}(\partial_\mu\phi^{abc})^2+\frac{g_1}{6!}\phi^{a_1b_1c_1}\phi^{a_1b_2c_2}\phi^{a_2b_1c_2}\phi^{a_3b_3c_1}\phi^{a_3b_2c_3}\phi^{a_2b_3c_3}\right)$$

- It is subchromatic and MST (erasing any color leaves the diagram connected).
- To obtain the large N solution it is convenient to rewrite



$$S=\int d^dx\left(\frac{1}{2}(\partial_\mu\phi^{abc})^2+\frac{\lambda}{3!}\phi^{a_1b_1c_1}\phi^{a_1b_2c_2}\phi^{a_2b_1c_2}\chi^{a_2b_2c_1}-\frac{1}{2}\chi^{abc}\chi^{abc}\right)$$

- Tensor counterpart of a bosonic SYK-like model.

  Murugan, Stanford, Witten

- The IR solution in general dimension:

$$3\Delta_\phi + \Delta_\chi = d \ , \qquad d/2 - 1 < \Delta_\phi < d/6$$

$$\frac{\Gamma(\Delta_\phi)\Gamma(d-\Delta_\phi)}{\Gamma(\frac{d}{2}-\Delta_\phi)\Gamma(-\frac{d}{2}+\Delta_\phi)} = 3\frac{\Gamma(3\Delta_\phi)\Gamma(d-3\Delta_\phi)}{\Gamma(\frac{d}{2}-3\Delta_\phi)\Gamma(-\frac{d}{2}+3\Delta_\phi)}$$

- In $d = 3 - \epsilon$

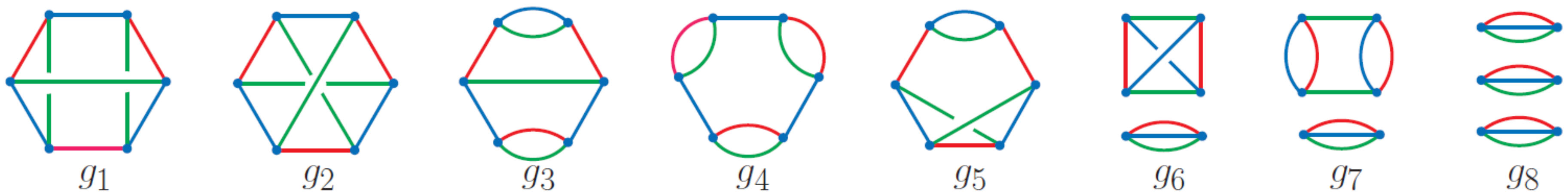
$$\Delta_\phi = \frac{1}{2} - \frac{\epsilon}{2} + \epsilon^2 - \frac{20\epsilon^3}{3} + \left(\frac{472}{9} + \frac{\pi^2}{3}\right)\epsilon^4 + \left(7\zeta(3) - \frac{12692}{27} - \frac{56\pi^2}{9}\right)\epsilon^5 + O\left(\epsilon^6\right)$$

- For d=2.9 find numerically

$$\Delta_\phi = 0.456264 \ , \qquad \Delta_\chi = 1.53121$$

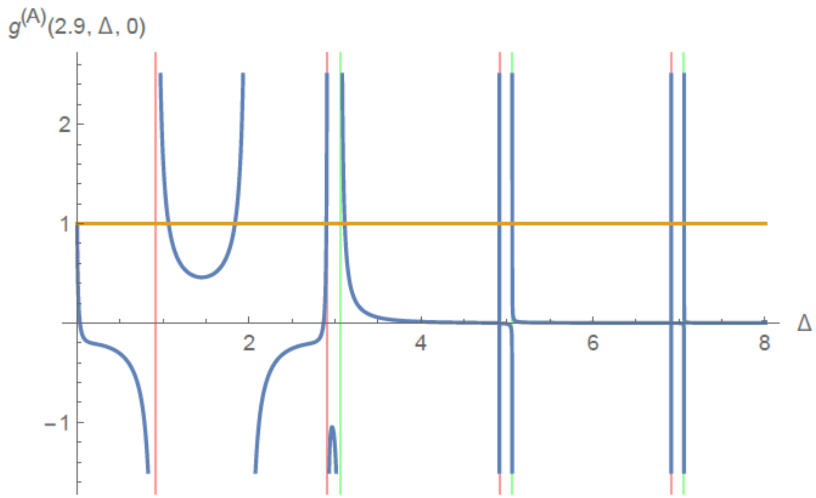
Finite N

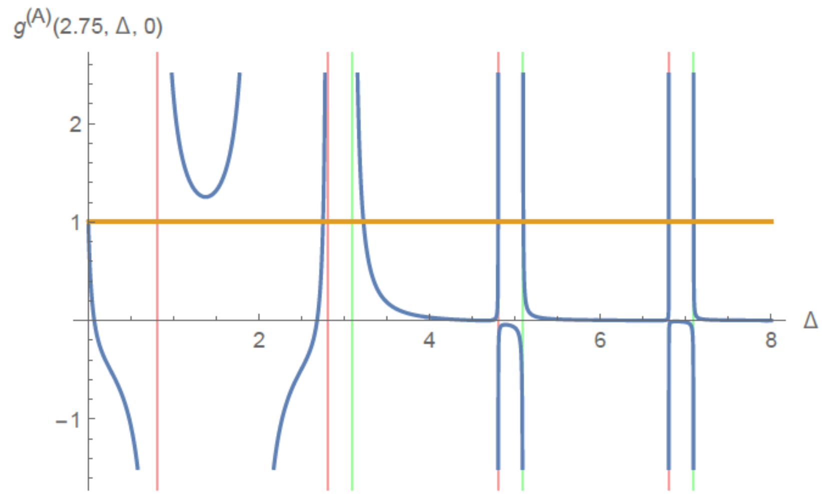
- The 3-ε expansion at finite N may be generated using standard perturbation theory.
- Need to include 7 more $O(N)^3$ invariant operators.



- The 8 beta functions have a "prismatic" fixed point" for N>53. At large N the scaling dimensions there agree with the Schwinger-Dyson results.

- Dimensions of bilinear operators in d=2.9 and 2.75





- The first root has expansion

$$\Delta_{\phi^2} = 1 - \epsilon + 32\epsilon^2 - \frac{976\epsilon^3}{3} + \left(\frac{30320}{9} + \frac{32\pi^2}{3}\right)\epsilon^4 + O\left(\epsilon^5\right)$$

- For $1.6799 < d < 2.8056$ Δ_{ϕ^2} becomes complex

$$\frac{d}{2} + i\alpha(d)$$

Complex CFTs

- May appear after two real fixed points have merged. Dymarsky, IK, Roiban; Rastelli, Pomoni; Kaplan, Lee, Son, Stephanov: Gorbenko, Rychkov, Zan; Grabner, Gromov, Kazakov, Korchemsky; …
- Correspond to (weakly) first-order transitions.
- Imaginary parts of scaling dimensions may be as small as

$$e^{-Nf(d)}$$

- Appear in the O(N) model in 4<d<6 due to the instanton effects McKane, Wallace, de Alcantara Bonfim (1984); Giombi, Huang, Klebanov, Pufu Tarnopolsky (2019)
- For large N an operator of dimension $\frac{d}{2}+i\alpha(d)$ corresponds in dual AdS to a scalar violating the Breitenlohner-Freedman stability bound:

$$m^2 < -d^2/4$$

Two-Flavor O(N)^3 Model

- Interaction of two rank-3 Majorana tensors with O(N)^3 symmetry, with a parameter α

Kim, IK, Tarnopolsky, Zhao

$$H = \frac{g}{4}\left(\psi_1^{a_1b_1c_1}\psi_1^{a_1b_2c_2}\psi_1^{a_2b_1c_2}\psi_1^{a_2b_2c_1} + \psi_2^{a_1b_1c_1}\psi_2^{a_1b_2c_2}\psi_2^{a_2b_1c_2}\psi_2^{a_2b_2c_1}\right)$$

$$+\frac{g\alpha}{2}\left(\psi_1^{a_1b_1c_1}\psi_1^{a_1b_2c_2}\psi_2^{a_2b_1c_2}\psi_2^{a_2b_2c_1} + \psi_1^{a_1b_1c_1}\psi_2^{a_1b_2c_2}\psi_1^{a_2b_1c_2}\psi_2^{a_2b_2c_1} + \psi_1^{a_1b_1c_1}\psi_2^{a_1b_2c_2}\psi_2^{a_2b_1c_2}\psi_1^{a_2b_2c_1}\right)$$

- Reduces to the bipartite model for $\alpha = -1$
- Melonic S-D equations give 2-pt function

$$G(t_2 - t_1) = -\left(\frac{1}{4\pi(3\alpha^2+1)g^2N^3}\right)^{\frac{1}{4}} \frac{\mathrm{sgn}(t_2 - t_1)}{|t_2 - t_1|^{1/2}}$$

Discrete Symmetries

- Particle-hole symmetry $\mathcal{P} = K \prod_I (\psi^I + \bar{\psi}^I)$

$$\psi^I = \frac{1}{\sqrt{2}}(\psi_1^I + i\psi_2^I) \ , \qquad \bar{\psi}^I = \frac{1}{\sqrt{2}}(\psi_1^I - i\psi_2^I)$$

$$KiK = -i \ , \qquad K\psi^I K = \psi^I \ , \qquad K\bar{\psi}^I K = \bar{\psi}^I$$

- The fermion number Q is conserved mod 4

$$Q = i\psi_1^I \psi_2^I = \frac{1}{2}[\bar{\psi}^I, \psi^I]$$

$$H = \frac{1}{4!} J_{IJKL} \left(\frac{1-3\alpha}{2} \left(\psi^I \psi^J \psi^K \psi^L + \bar{\psi}^I \bar{\psi}^J \bar{\psi}^K \bar{\psi}^L \right) + 3(1+\alpha) \bar{\psi}^I \bar{\psi}^J \psi^K \psi^L \right)$$

Bilinear Operators

- The scaling dimensions of

$$O_1^{2n+1} = \psi_1 \partial_\tau^{2n+1} \psi_1 + \psi_2 \partial_\tau^{2n+1} \psi_2 \ , \qquad O_2^{2n+1} = \psi_1 \partial_\tau^{2n+1} \psi_1 - \psi_2 \partial_\tau^{2n+1} \psi_2 \ ,$$
$$O_3^{2n+1} = \psi_1 \partial_\tau^{2n+1} \psi_2 + \psi_2 \partial_\tau^{2n+1} \psi_1 \ , \qquad O_4^{2n} = \psi_1 \partial_\tau^{2n} \psi_2 - \psi_2 \partial_\tau^{2n} \psi_1 \ ,$$

are determined from

$$(g_1(h), g_2(h), g_3(h), g_4(h)) = \left(g_A(h), \frac{1-\alpha^2}{1+3\alpha^2} g_A(h), \frac{2\alpha(1+\alpha)}{1+3\alpha^2} g_A(h), \frac{6\alpha(1-\alpha)}{1+3\alpha^2} g_S(h) \right)$$

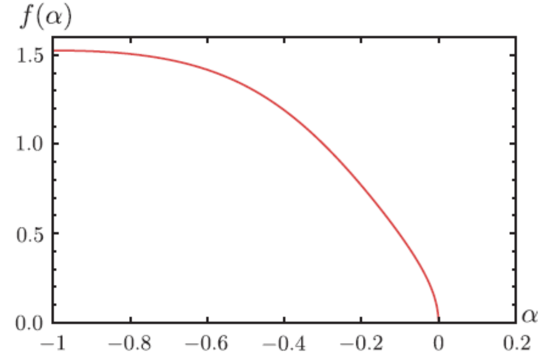
$$g_A(h) = -\frac{3}{2} \frac{\tan(\frac{\pi}{2}(h-\frac{1}{2}))}{h-1/2} \qquad g_S(h) = -\frac{1}{2} \frac{\tan(\frac{\pi}{2}(h+\frac{1}{2}))}{h-1/2}$$

Duality

- Use transformation $\psi_1 = \frac{1}{\sqrt{2}}(\tilde{\psi}_1 + \tilde{\psi}_2),\ \psi_2 = \frac{1}{\sqrt{2}}(\tilde{\psi}_1 - \tilde{\psi}_2)$
- Find equivalence $(g,\ \alpha) \sim (g',\ \alpha')$

$$g' = \frac{(3\alpha + 1)g}{2} \qquad \alpha' = \frac{-\alpha + 1}{3\alpha + 1}$$

- Apart from overall scaling of energies, can restrict to $-1 \leq \alpha \leq \frac{1}{3}$
- For $\alpha < 0$ operator Q has complex dimension $\frac{1}{2} \pm i f(\alpha)$ where $f \tanh(\pi f/2) = \frac{3\alpha^2 - 3\alpha}{3\alpha^2 + 1}$
- For small α



$$f(\alpha) = \sqrt{\frac{-6\alpha}{\pi}}(1 + O(\alpha))$$

Coupled SYK Models

- To study low-energy spectrum numerically, replace the two-flavor tensor model by its SYK counterpart.
- Double SYK model with a quartic coupling

$$H = \frac{1}{4!} J_{ijkl} \left(\chi_1^i \chi_1^j \chi_1^k \chi_1^l + \chi_2^i \chi_2^j \chi_2^k \chi_2^l + 6\alpha \chi_1^i \chi_1^j \chi_2^k \chi_2^l \right)$$

- A generalization of the Gross-Rosenhaus two-flavor model.
- Gives the same large N S-D equations and scaling dimensions as the tensor model.

Green's Functions

- Allow for both diagonal and off diagonal:

$$G_{ab}(\tau,\tau') = \frac{1}{N_{\text{SYK}}}\langle T\chi_a^i(\tau)\chi_b^i(\tau')\rangle$$

- General properties

$$G_{11}(\tau) = -G_{11}(-\tau)\ , \qquad G_{22}(\tau) = -G_{22}(-\tau)\ , \qquad G_{12}(\tau) = -G_{21}(-\tau)$$

- Additional discrete symmetry implies

$$G_{12}(-\tau) = -G_{21}(\tau) = G_{12}(\tau)\ , \qquad G_{22}(\tau) = G_{11}(\tau)$$

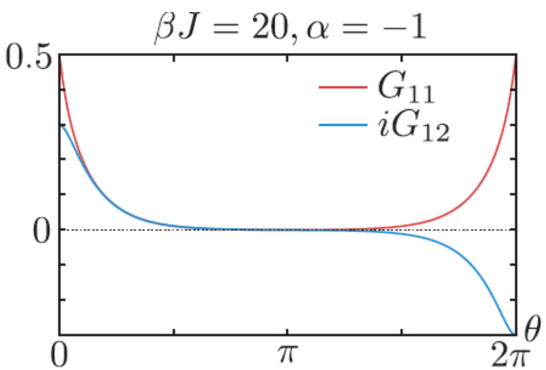
$$-\frac{\beta S_{\text{eff}}}{N_{\text{SYK}}} = \log \text{Pf}(\delta_{ab}\partial_\tau - \Sigma_{ab}) - \beta\int_0^\beta d\tau\Big(\Sigma_{11}G_{11} + \Sigma_{12}G_{12} - \frac{J^2}{4}\big((1+3\alpha^2)(G_{11}^4 + G_{12}^4) + 12\alpha(1-\alpha)G_{11}^2G_{12}^2\big)\Big)$$

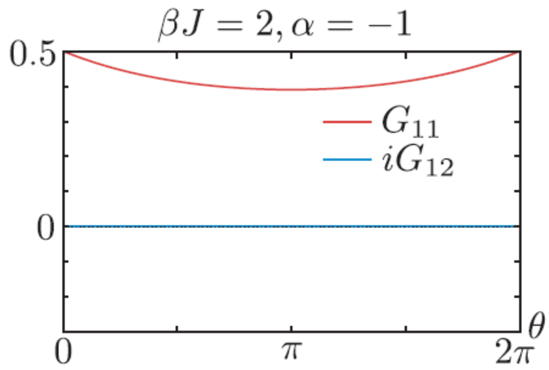
- The Schwinger-Dyson equations become

$$\partial_\tau G_{11}(\tau) - \int d\tau' \big(\Sigma_{11}(\tau-\tau')G_{11}(\tau') - \Sigma_{12}(\tau-\tau')G_{12}(\tau')\big) = \delta(\tau)$$
$$\partial_\tau G_{12}(\tau) - \int d\tau' \big(\Sigma_{11}(\tau-\tau')G_{12}(\tau') + \Sigma_{12}(\tau-\tau')G_{11}(\tau')\big) = 0 \ ,$$

$$J^{-2}\Sigma_{11}(\tau) = (1+3\alpha^2)G_{11}^3(\tau) + 6\alpha(1-\alpha)G_{11}(\tau)G_{12}^2(\tau)$$
$$J^{-2}\Sigma_{12}(\tau) = (1+3\alpha^2)G_{12}^3(\tau) + 6\alpha(1-\alpha)G_{11}^2(\tau)G_{12}(\tau)$$

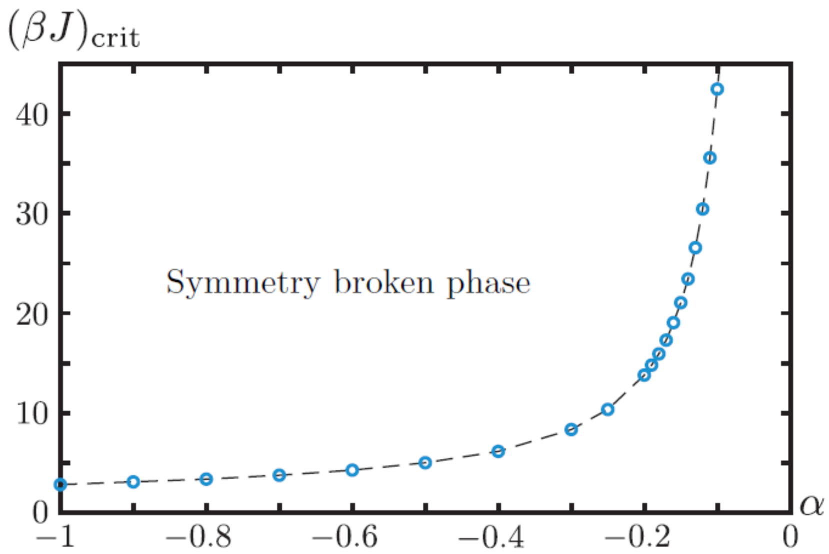
- Solutions exhibit a phase transition





(Nearly) Conformal Window

- The critical temperature depends on alpha

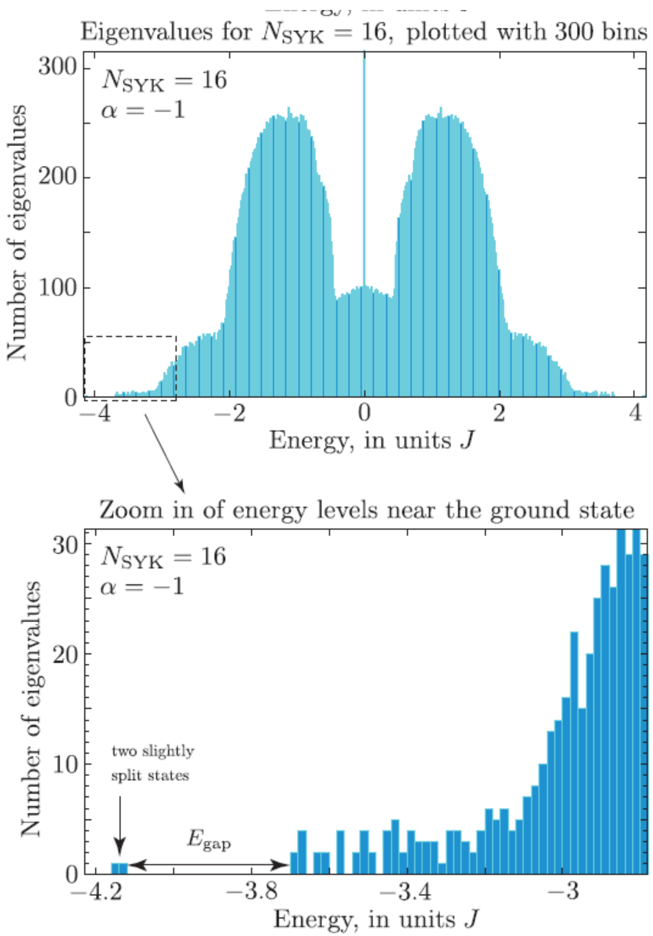


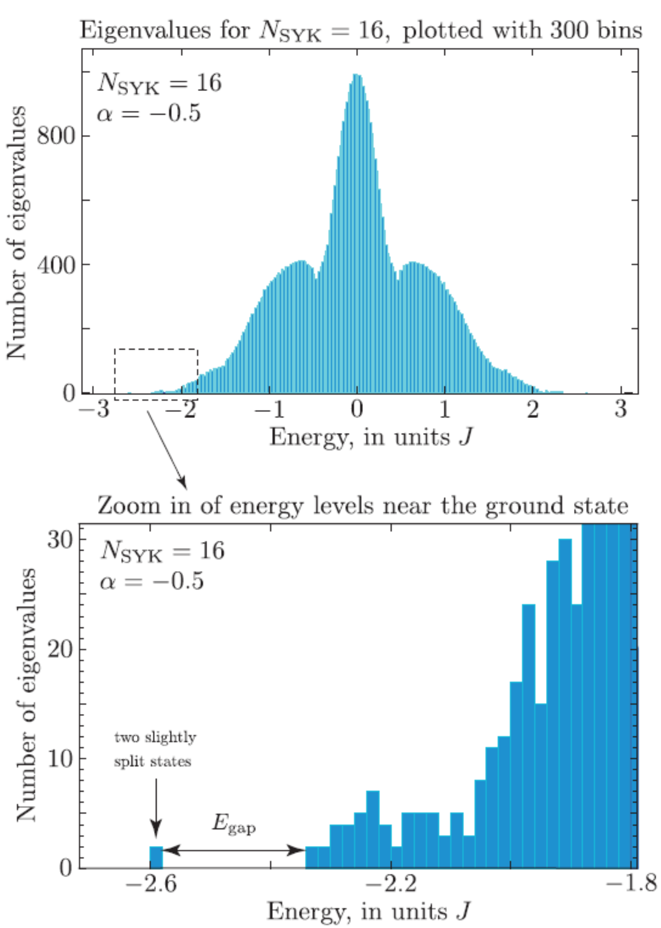
- For $0 \leq \alpha \leq 1/3$ there is no symmetry breaking.
- The low-T entropy does not vary (g-theorem)

$$S_0 = 2c_0 N \ , \qquad c_0 = \frac{1}{8}\log 2 + \frac{K}{2\pi} \approx 0.2324$$

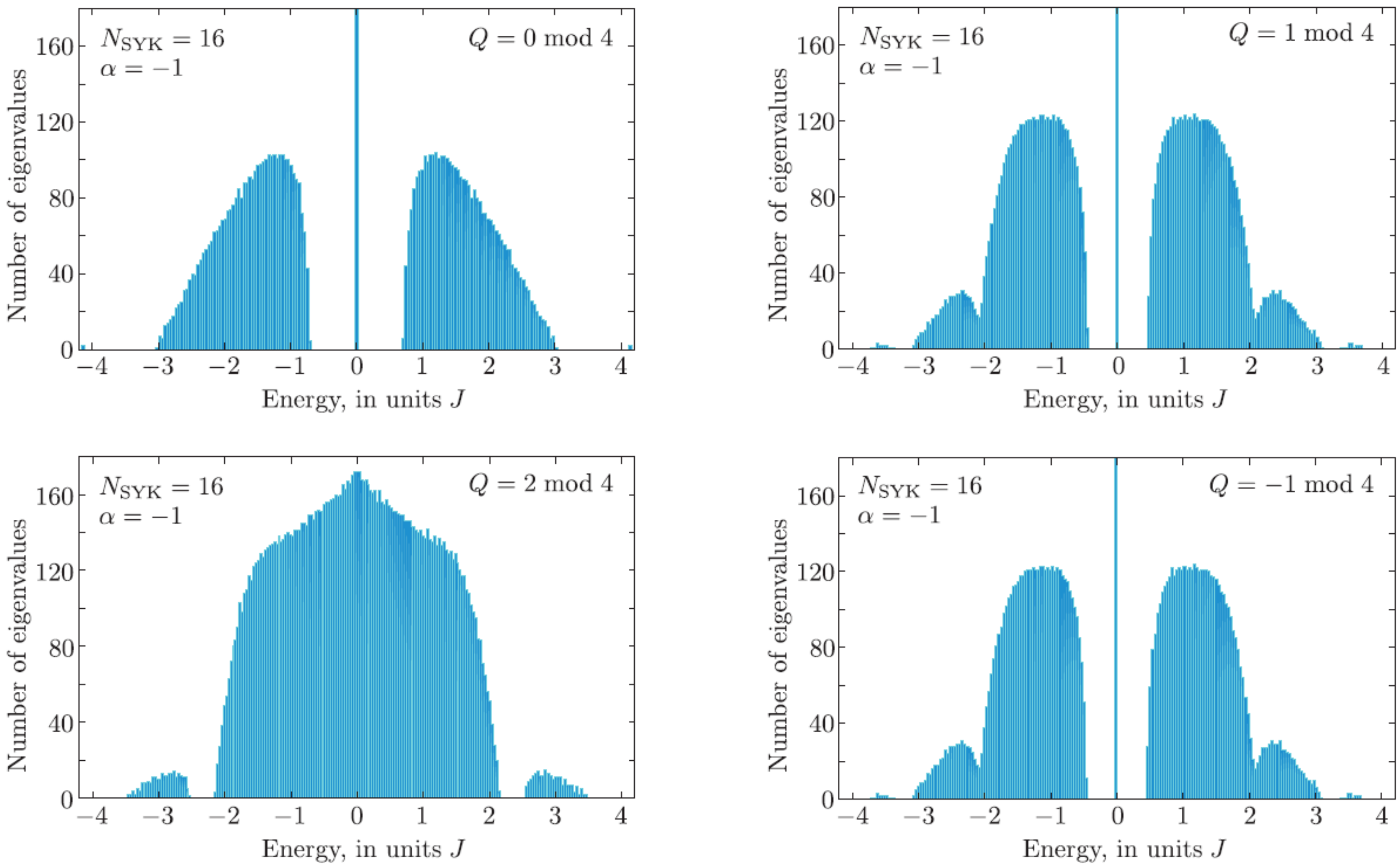
Exact Diagonalizations

- When $-1 \leq \alpha < 0$ there are two nearly degenerate lowest states followed by a gap:



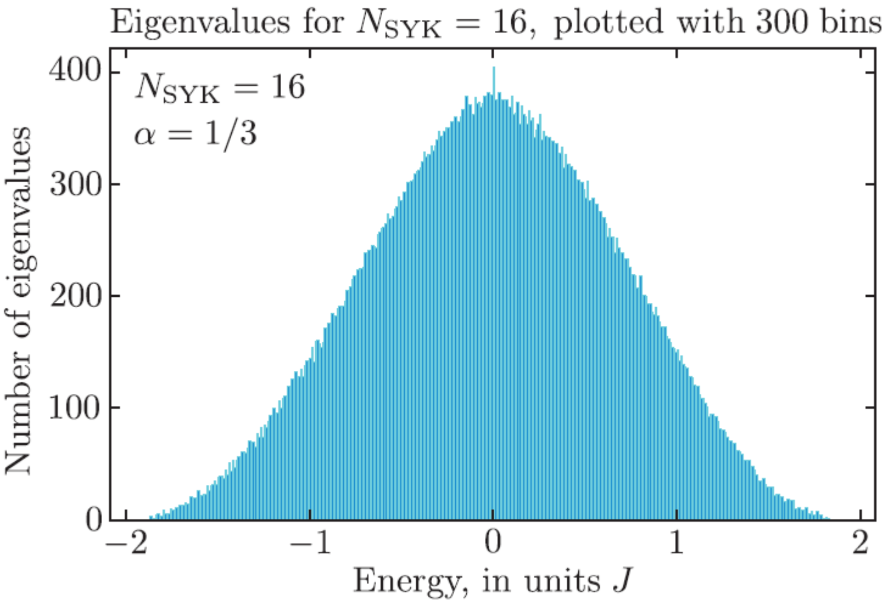


- The spectrum breaks up into 4 sectors due to the conservation of Q mod 4.



Symmetry Breaking

- The two lowest states show the breaking of Z_2 symmetry for $-1 \leq \alpha < 0$
- For $0 \leq \alpha \leq 1/3$ there is no gap.
- No symmetry breaking:

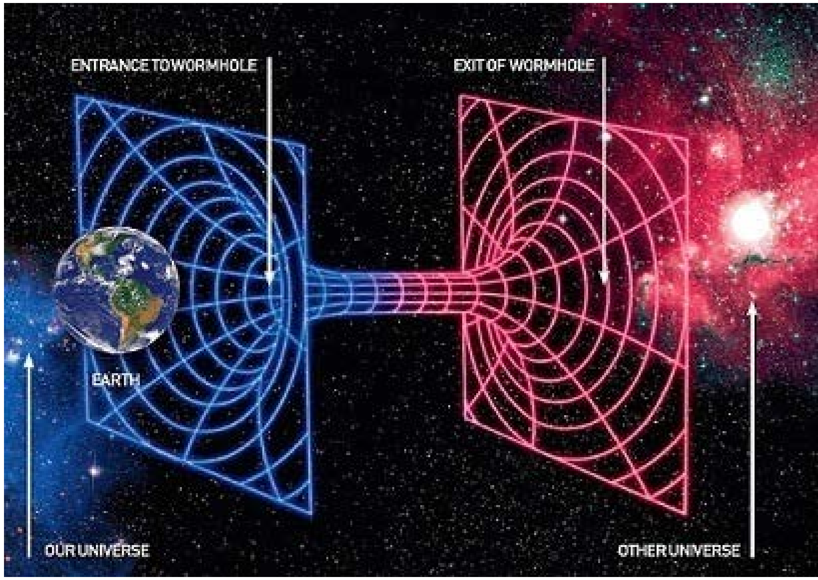


Dual of a Wormhole?

- We can interpret the gapped phase with small low-T entropy as the dual of a traversable wormhole geometry. Maldacena, Qi
- It appears only for one sign of the coupling:

$$\alpha < 0$$

- As in the Gao-Jafferis-Wall model.



Conclusions

- The $O(N)^3$ fermionic tensor quantum mechanics seems to be the closest non-random counterpart of the basic SYK model for Majorana fermions.
- Solution of S-D equations indicates a (nearly) conformal phase with real scaling dimenions.
- Bosonic or fermionic generalizations can lead to formally complex scaling dimensions with real part d/2, indicating an instability of the conformal phase.

- Studied quantum mechanics of two rank-3 Majorana tensors with $O(N)^3$ symmetry and quartic terms coupling the two, and its SYK counterpart.
- Find a “(nearly) conformal window” for positive coupling.
- A complex scaling dimension appears only for negative coupling, where the true low T phase exhibits the symmetry breaking. Numerical calculations also indicate a gapped spectrum.
- Relation to the Gao-Jafferis-Wall wormhole construction?
- Physical applications?