Hidden Symmetries near Cosmological Singularities

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Genericity of Cosmological Singularities?

Landau 1959: Is the big bang singularity of Friedmann universes a generic property of general relativistic cosmologies, or is it an artefact of the high degree of symmetry of these solutions?

Khalatnikov and Lifshitz 1963: look for generic inhomogeneous and anisotropic solutions near a singularity

$$ds^2 = -dt^2 + (a^2 \ell_i \ell_j + b^2 m_i m_j + c^2 n_i n_j) dx^i dx^j$$

single homogeneous Friedmann scale factor $a(t) \to$ three inhomogeneous scale factors $a(t, \mathbf{x})$, $b(t, \mathbf{x})$, $c(t, \mathbf{x})$

KL63 did not succeed in finding the "general" solution of the complicated, coupled dynamics of a, b, c and tentatively concluded that a singularity is not generic.

Genericity of Cosmological Singularities?

local collapse: Penrose 1965; cosmology: Hawking 1966-7, Hawking-Penrose 1970: Theorems about genericity of cosmological "singularity".

They prove generic "incompleteness" of spacetime, without giving any information about the "singularity".

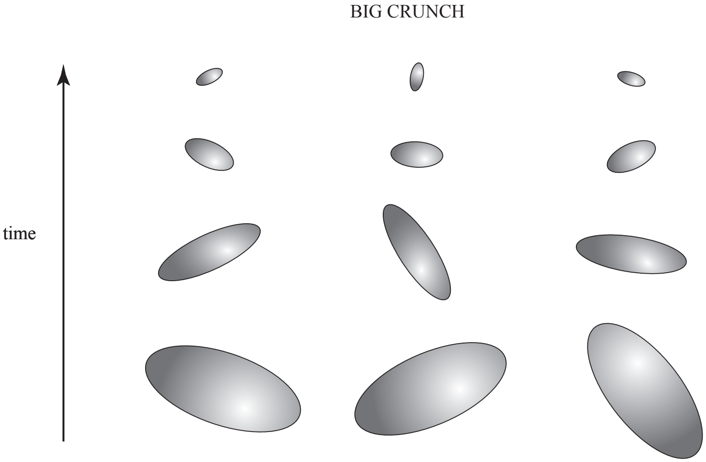
Belinsky, Khalatnikov, Lifshitz 1969:

• introduce a new approach to construct the "general" solution near $abc \to 0$ of the coupled (inhomogeneous) dynamics of $a(t, \mathbf{x})$, $b(t, \mathbf{x})$, $c(t, \mathbf{x})$,

• find that, at each point of space $\mathbf{x}$, the dynamics of a, b, c is chaotic.

The BKL conjecture has been confirmed both by numerical simulations (Weaver-Isenberg-Berger 1998, Berger-Moncrief 1998, Berger et al 1998-2001; Garfinkle 2002-2007; Berger's Living Review) and by analytical studies (Damour-Henneaux-Nicolai 2003; Uggla et al 2003-2007; Damour-De Buyl 2008).

BKL chaos near a big bang or a big crunch



Dynamics of BKL a, b, c system

January 1968, at the Institut Henri Poincaré, Isaak Khalatnikov gives a seminar in which he announces to the western world the results of BKL. He shows the system of equations for the three local scale factors a, b, c [with new time variable $d\tau = -dt/(abc)$]

$$2\frac{d^2 \ln a}{d\tau^2} = (b^2 - c^2)^2 - a^4$$

$$2\frac{d^2 \ln b}{d\tau^2} = (c^2 - a^2)^2 - b^4$$

$$2\frac{d^2 \ln c}{d\tau^2} = (a^2 - b^2)^2 - c^4$$

J.A. Wheeler was in the audience and immediately pointed out the possibility of a mechanical analogy for this model. He informed his former student Charles Misner (who was independently working on the Bianchi IX dynamics) of the BKL results. In 1969 Misner published a mechanical-like, Lagrangian analysis of the Bianchi IX (a, b, c) system under the catchy name of "mixmaster universe".

Cosmological Billiards

(Misner 1969a, 1969b [quantum], Chitre 1972, . . ., Damour-Henneaux-Nicolai 2003, . . ., Belinski-Henneaux 2018)

$$ds^2 = -dt^2 + (a^2 \ell_i \ell_j + b^2 m_i m_j + c^2 n_i n_j) dx^i dx^j$$

exponential parametrisation: $a = e^{-\beta^1}$, $b = e^{-\beta^2}$, $c = e^{-\beta^3}$

Lagrangian ruling the dynamics of the β's at each spatial point

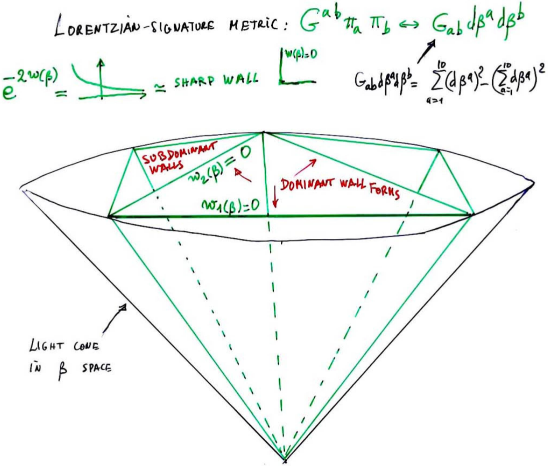
$$\mathcal{L} = \frac{1}{2} G_{ab} \dot{\beta}^a \dot{\beta}^b - V(\beta)$$

Kinetic metric $G_{ab} \dot{\beta}^a \dot{\beta}^b = \sum_a (\dot{\beta}^a)^2 - \left(\sum_a \dot{\beta}^a\right)^2$ (DeWitt metric)

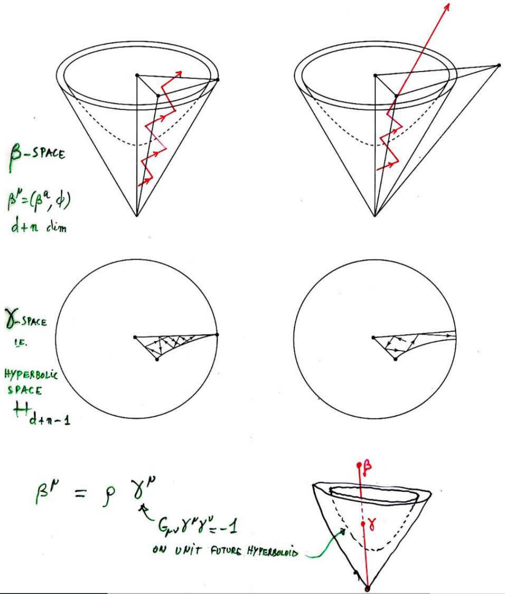
Potential $V(\beta) = \sum_a c_A(\ldots)\, e^{-2w_A(\beta)}$

Wall forms $w_A(\beta)$: e.g. gravitational walls: $w^{(g)}_{abc}(\beta) = \sum_e \beta^e + \beta^a - \beta^b - \beta^c$

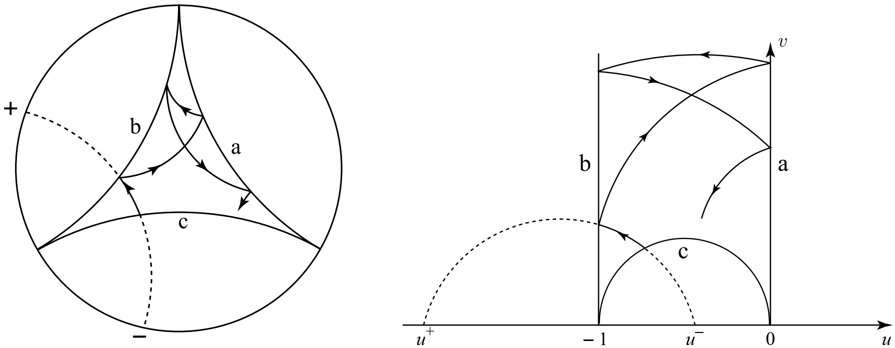
Billiard in β space: Toda-like exponential potentials $V(\beta) = \sum_a c_A(\ldots)\, e^{-2 w_A(\beta)}$



Einstein Billiards (chaotic versus non-chaotic)



Chaotic billiard for $D = 4$ gravity (BKL, Misner, Chitre)



Non-chaotic Billiards

Asymptotically Kasner-like; amenable to Fuchsian analysis if one assumes analyticity in space

$D = 4$ gravity $+$ scalar field (Belinsky-Khalatnikov 73, Andersson-Rendall 01)

$D \geq 11$ pure gravity (Demaret et al 85, Damour-Henneaux-Rendall-Weaver 02)

$D \geq 39$ pure gravity, but without assuming analyticity: Rodnianski-Speck 18 gives a mathematical proof for near-isotropic initial data.

Kac-Moody algebras

Generalization of the well-known "triangular" structure of $A_1 = so(3) = su(2) = sl(2)$: diagonalizable (Cartan) generator: J_z, and raising/lowering generators: $J_\pm = J_x \pm i\, J_y$ with $[J_z, J_+] = +J_+ \,; \quad [J_z, J_-] = -J_- \,; \quad [J_+, J_-] = 2\, J_z$

Rank r: r mutually commuting Cartan generators h_i and r simple raising (e_i) and lowering (f_i) generators:

$$[h_i, h_j] = 0\,; [h_i, e_j] = A_{ij}\, e_j\,; \quad [h_i, f_j] = -A_{ij}\, f_j\,; \quad [e_i, f_j] = \delta_{ij}\, h_j$$

Serre relations: $ad_{e_i}^{1-A_{ij}}\, e_j = 0$; $ad_{f_i}^{1-A_{ij}}\, f_j = 0$

A_{ij} = Cartan matrix: $A_{ii} = +2$, $A_{ij} \in -\mathbb{N}$

Roots: α = linear form on Cartan: $h = \sum_i \beta^i h_i \to \alpha(h) = \alpha_i \beta^i$

$$E_\alpha \sim [e_{i_1}[e_{i_2}[e_{i_3}, \ldots]]] \qquad \alpha = n_1 \alpha^{(1)} + n_2 \alpha^{(2)} + \ldots + n_r \alpha^{(r)}$$

$e_i = E_{\alpha^{(i)}}$ simple roots; $\quad \left[h, E_\alpha^{(s)}\right] = \alpha(h)\, E_\alpha^{(s)} \qquad A_{ij} = \frac{2(\alpha^{(i)}, \alpha^{(j)})}{(\alpha^{(i)}, \alpha^{(i)})}$

Dynkin Diagrams ($=$ Cartan Matrix) of E_{10} and AE_3

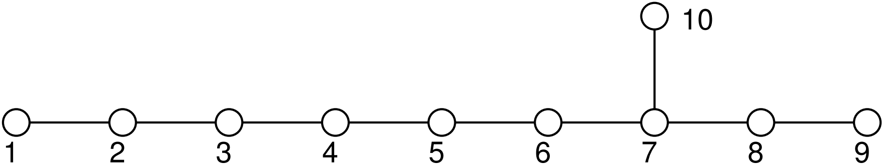
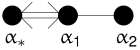


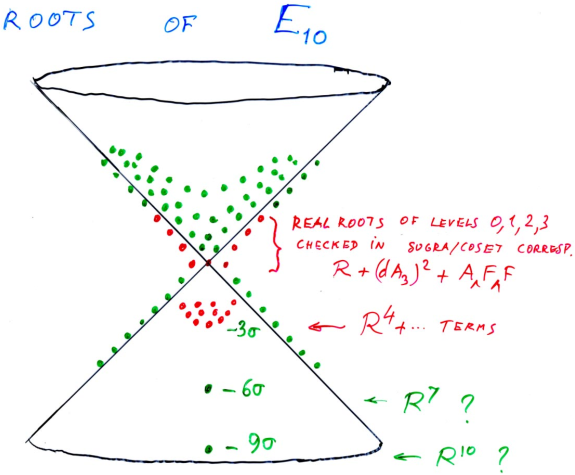
Figure: *Dynkin diagram of E_{10} with numbering of nodes.*

Cartan matrix of AE_3: $(A_{ij}) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$



Dynkin diagram AE_3

E_{10} and AE_3 Root Diagrams: each root $\alpha \leftrightarrow$ a Lie-algebra generator E_α



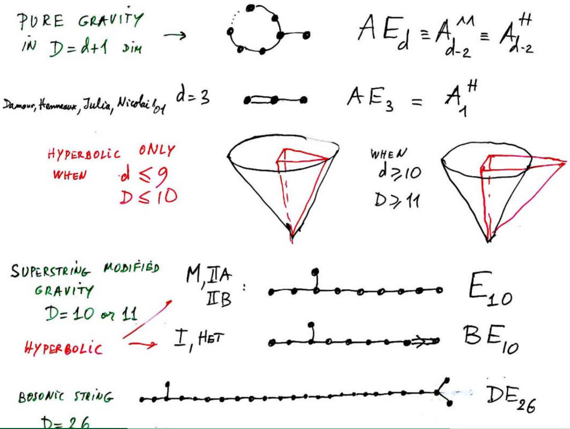
Cosmological Singularities and Hyperbolic Kac-Moody Algebras:

Billiard Walls = Kac-Moody Roots

potentials $V(\beta) = \sum_a c_A(\ldots)\, e^{-2w_A(\beta)}$ **with** $w_A(\beta) = \alpha(\beta)$

$+$ **much deeper gravity/coset correspondence**

Damour, Henneaux 2001; Damour, Henneaux, Julia, Nicolai 2001; Damour, Henneaux, Nicolai 2002



Bosonic EOM of SUGRA_{11}

$D = 11$ spacetime, zero-shift slicing ($N^i = 0$) time-independent spatial coframe $\theta^a(x) \equiv E^a{}_i(x)dx^i$, $i = 1, \dots, 10$; $a = 1, \dots, 10$ choose time coordinate x^0 s.t. lapse $N = \sqrt{G}$ with $G := \det G_{ab}$

structure constants of frame: $d\theta^a = \frac{1}{2} C^a{}_{bc}\, \theta^b \wedge \theta^c$; frame derivative $\partial_a \equiv E^i{}_a(x)\partial_i$; 3-form $\mathcal{A}$; 4-form $\mathcal{F} = d\mathcal{A}$; $2G_{ad}\Gamma^d_{bc} = C_{abc} + C_{bca} - C_{cab} + \partial_b G_{ca} + \partial_c G_{ab} - \partial_a G_{bc}$

$$ds^2 = -N^2(dx^0)^2 + G_{ab}\theta^a\theta^b$$

$$\mathcal{F} = \frac{1}{3!}\mathcal{F}_{0abc}dx^0 \wedge \theta^a \wedge \theta^b \wedge \theta^c + \frac{1}{4!}\mathcal{F}_{abcd}\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d$$

$$\partial_0(G^{ac}\partial_0 G_{cb}) = \frac{1}{6}G\mathcal{F}^{a\beta\gamma\delta}\mathcal{F}_{b\beta\gamma\delta} - \frac{1}{72}G\mathcal{F}^{\alpha\beta\gamma\delta}\mathcal{F}_{\alpha\beta\gamma\delta}\delta^a_b - 2GR^a_b(\Gamma, C)$$

$$\begin{aligned}\partial_0(G\mathcal{F}^{0abc}) &= \frac{1}{144}\varepsilon^{abca_1a_2a_3b_1b_2b_3b_4}\mathcal{F}_{0a_1a_2a_3}\mathcal{F}_{b_1b_2b_3b_4} \\ &+ \frac{3}{2}G\mathcal{F}^{de[ab}C^{c]}{}_{de} - GC^e{}_{de}\mathcal{F}^{dabc} - \partial_d(G\mathcal{F}^{dabc})\end{aligned}$$

$$\partial_0\mathcal{F}_{abcd} = 6\mathcal{F}_{0e[ab}C^e{}_{cd]} + 4\,\partial_{[a}\mathcal{F}_{0bcd]}$$

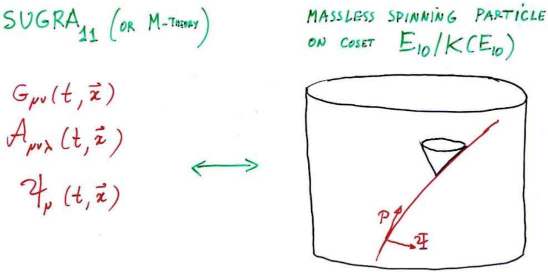
Gravity/Kac-Moody Coset Correspondence

Appearance of E_{10} in the “near cosmological singularity limit” (where a Belinski-Khalatnikov-Lifshitz chaotic behavior arises) suggests the existence of a supergravity/E_{10} coset correspondence (Damour, Henneaux, Nicolai ′02)

[related suggestions: E_{10}, Ganor ′99 ′04; E_{11} : West ′01]

The ‘singularity’ is ‘resolved’ by the effective ‘disappearance’ of space, and the replacement of dynamical fields, $g_{ij}(t, \mathbf{x}), \mathcal{A}_{ijk}(t, \mathbf{x}), \ldots$ by a Lie-algebraic variable $g(t) \in E_{10}(\mathbb{Z}) \backslash E_{10}(\mathbb{R}) / K_{10}(\mathbb{R})$

Basic Idea: two ‘dual’ descriptions



Supergravity Description	Coset Description
$G_{\mu\nu}(t, \mathbf{x}), \mathcal{A}_{\mu\nu\lambda}(t, \mathbf{x}), \psi_\mu(t, \mathbf{x})$ in (T^{10} ?) compactified space	$g(t) \in E_{10}(\mathbb{Z})\backslash E_{10}(\mathbb{R})/K_{10}(\mathbb{R})$
$\mathcal{R} \ll \ell_P^{-2}$	$\mathcal{R} \gg \ell_P^{-2}$

Gravity/coset correspondence

(super)gravity $\leftrightarrow$ massless (spinning) particle on G/K

$g(t) \in G/K$; velocity $v \equiv \partial_t g\, g^{-1} \in \mathrm{Lie}(G)$ is decomposed into $v = \mathcal{P} + \mathcal{Q}$ where $\mathcal{Q} \in \mathrm{Lie}(K)$ and $\mathcal{P} = v^{\mathrm{sym}} = \frac{1}{2}(v + v^T) \in \mathrm{Lie}(G) - \mathrm{Lie}(K)$

Coset Action for massless particle:

$$S^{\mathrm{coset}}_{1_{\mathrm{BOS}}} = \int \frac{dt}{n(t)}\, \frac{1}{4}\, \langle \mathcal{P}(t), \mathcal{P}(t) \rangle$$

$n(t)$: coset lapse $\rightarrow$ constraint $\langle \mathcal{P}(t), \mathcal{P}(t) \rangle = 0$

For hyperbolic (or more generally Lorentzian) Kac-Moody algebras the coset G/K is an infinite dimensional Lorentzian space of signature $- + + + + + \ldots$

Evidence for gravity/coset correspondence

Damour, Henneaux, Nicolai 02; Damour, Kleinschmidt, Nicolai 06; de Buyl, Henneaux, Paulot 06; Kleinschmidt, Nicolai 06

Insert in $S_1^{\mathrm{COSET}} = \int dt \{ \frac{1}{4n(t)} \langle P(t), P(t) \rangle - \frac{i}{2} (\Psi(t) \mid \mathcal{D}^{\mathrm{vs}} \Psi(t))_{\mathrm{vs}} + \ldots \}$ the $GL(10)$ level expansion of the coset element

$$g(t) = \exp(h_b^a(t)\, K_a^b) \times$$

$$\times \exp \left[\frac{1}{3!} A_{abc}(t)\, E^{abc} + \frac{1}{6!} A_{a_1 \ldots a_6}(t) E^{a_1 \ldots a_6} + \frac{1}{9!} A_{a_0 | a_1 \ldots a_8}(t)\, E^{a_0 | a_1 \ldots a_8} + \ldots \right].$$

Agreement (up to height 29) of EOM of $g^{ab}(t) = (e^h)^a_c (e^h)^b_c$, $A_{abc}(t)$, $A_{a_1 \ldots a_6}(t)$, $A_{a_0|a_1 \ldots a_8}(t)$, and $\Psi_a^{\mathrm{coset}}(t)$ with supergravity EOM (including lowest spatial gradients) for $G_{\mu\nu}(t, \mathbf{x}), \mathcal{A}_{\mu\nu\lambda}(t, \mathbf{x}), \psi_\mu(t, \mathbf{x})$ with dictionary:

$g^{ab}(t) = G^{ab}(t, \mathbf{x}_0)$, $\qquad \dot{A}_{abc}(t) = \mathcal{F}_{0abc}(t, \mathbf{x}_0)$,

$DA^{a_1 \ldots a_6}(t) = -\frac{1}{4!} \varepsilon^{a_1 \ldots a_6 b_1 \ldots b_4} \mathcal{F}_{b_1 \ldots b_4}(t, \mathbf{x}_0)$,

$DA^{b|a_1 \ldots a_8}(t) = \frac{3}{2} \varepsilon^{a_1 \ldots a_8 b_1 b_2} C^b_{b_1 b_2}(t, \mathbf{x}_0)$

and $\Psi_a^{\mathrm{coset}}(t) = G^{1/4} \psi_a(t, \mathbf{x}_0)$

Moreover, $\exists$ roots in E_{10} formally associated with the infinite towers of higher spatial gradients of $G_{\mu\nu}(t, \mathbf{x}), \mathcal{A}_{\mu\nu\lambda}(t, \mathbf{x}), \psi_\mu(t, \mathbf{x})$

$K(E_{10})$ Structure of Gravitino Eq. of Motion

In the gauge $\psi_0^{(11)} = \Gamma_0\Gamma^a\psi_a^{(11)}$, the equation of motion of the rescaled gravitino $\psi_a^{(10)} := g^{1/4}\psi_a^{(11)}$ (neglecting cubic terms) reads

$$
\begin{aligned}
\mathcal{E}_a &= \partial_t\psi_a^{(10)} + \omega_{t\,ab}^{(11)}\psi^{(10)b} + \frac{1}{4}\omega_{t\,cd}^{(11)}\Gamma^{cd}\psi_a^{(10)} \\
&- \frac{1}{12}F_{tbcd}^{(11)}\Gamma^{bcd}\psi_a^{(10)} - \frac{2}{3}F_{tabc}^{(11)}\Gamma^b\psi^{(10)c} + \frac{1}{6}F_{tbcd}^{(11)}\Gamma_a{}^{bc}\psi^{(10)d} \\
&+ \frac{N}{144}F_{bcde}^{(11)}\Gamma^0\Gamma^{bcde}\psi_a^{(10)} + \frac{N}{9}F_{abcd}^{(11)}\Gamma^0\Gamma^{bcde}\psi_e^{(10)} - \frac{N}{72}F_{bcde}^{(11)}\Gamma^0\Gamma_{abcdef}\psi^{(10)f} \\
&+ N(\omega_{a\,bc}^{(11)} - \omega_{b\,ac}^{(11)})\Gamma^0\Gamma^b\psi^{(10)c} + \frac{N}{2}\omega_{a\,bc}^{(11)}\Gamma^0\Gamma^{bcd}\psi_d^{(10)} - \frac{N}{4}\omega_{b\,cd}^{(11)}\Gamma^0\Gamma^{bcd}\psi_a^{(10)} \\
&+ Ng^{1/4}\Gamma^0\Gamma^b\Big(2\partial_a\psi_b^{(11)} - \partial_b\psi_a^{(11)} - \frac{1}{2}\omega_{c\,cb}^{(11)}\psi_a^{(11)} - \omega_{0\,0a}^{(11)}\psi_b^{(11)} + \frac{1}{2}\omega_{0\,0b}^{(11)}\psi_a^{(11)}\Big).
\end{aligned}
$$

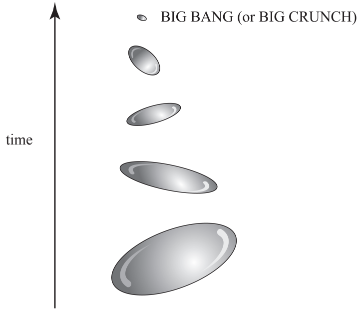
Apart from the last line, this is equivalent to the $K(E_{10})$-covariant equation

$$
0 = \overset{\text{vs}}{\mathcal{D}}\Psi(t) := \Big(\partial_t - \overset{\text{vs}}{\mathcal{Q}}(t)\Big)\Psi(t).
$$

expressing the parallel propagation of the $K(E_{10})$ vector-spinor $\Psi(t)$ along the $E_{10}/K(E_{10})$ worldline of the coset particle, with the $K(E_{10})$ connection $\mathcal{Q}(t) := \frac{1}{2}(v(t) - v^T(t)) \in \mathrm{Lie}(\mathrm{K}(\mathrm{E}_{10}))$, with $v(t) = \partial_t g g^{-1} \in \mathfrak{e}_{10} \equiv \mathrm{Lie}(E_{10})$.

A concrete case study (Damour, Spindel 2013, 2014, 2017)

• Quantum supersymmetric Bianchi IX, i.e. quantum dynamics of a supersymmetric triaxially squashed three-sphere: with squashing parameters $a = e^{-\beta^1}, \quad b = e^{-\beta^2}, \quad c = e^{-\beta^3}$



Susy Quantum Cosmology: Obregon et al $\geq$ 1990; D'Eath, Hawking, Obregon, 1993, D'Eath $\geq$ 1993, Csordas, Graham 1995, Moniz $\geq$ 94, ...

Quantum susy Bianchi IX

Technically: Reduction to one, time-like, dimension of the action of $D = 4$ simple supergravity for an $SU(2)$-homogeneous (Bianchi IX) cosmological model $\rightarrow$ (essentially) Susy Quantum Mechanical model

$$
\begin{aligned}
g_{\mu\nu}\, dx^\mu\, dx^\nu &= -N^2(t) dt^2 \\
&+ g_{ab}(t)(\tau^a(x) + N^a(t)dt)(\tau^b(x) + N^b(t)dt)\,,
\end{aligned}
$$

τ^a: left-invariant one-forms on $SU(2) \approx S_3 : d\tau^a = \frac{1}{2} C^a_{bc}\, \tau^b \wedge \tau^c$; here $C^a_{bc} = \varepsilon_{abc}$ plays the role of a nonabelian “gravitational flux”, or constant momentum of (coset) dual graviton.

Dynamical degrees of freedom

• 6 bosonic dof: Gauss-decomposition of the metric: $g_{bc} = \sum_{\hat{a}=1}^{3} e^{-2\beta^a} S^{\hat{a}}{}_b(\varphi_1, \varphi_2, \varphi_3)\, S^{\hat{a}}{}_c(\varphi_1, \varphi_2, \varphi_3)$
$\beta^a = (\beta^1(t), \beta^2(t), \beta^3(t))$ cologarithms of the squashing parameters a, b, c of 3-sphere $a = e^{-\beta^1}$, $b = e^{-\beta^2}$, $c = e^{-\beta^3}$ and three Euler angles: $\varphi_a = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$ parametrizing the orthogonal matrix $S^{\hat{a}}_b$

• and 12 fermionic dof: Gravitino components in specific gauge-fixed orthonormal frame $\theta^{\hat{\alpha}}$ canonically associated with the Gauss-decomposition $\theta^{\hat{0}} = N(t)dt$, $\theta^{\hat{a}} = \sum_b e^{-\beta^a(t)} S^{\hat{a}}{}_b(\varphi_c(t))(\tau^b(x) + N^b(t)dt)$

• redefinitions of the gravitino field:

$$\Psi^A_{\hat{\alpha}}(t) := g^{1/4}\, \psi^A_{\hat{\alpha}} \quad \text{and} \quad \Phi^a_A := \Sigma_B\, \gamma^{\hat{a}}_{AB}\, \Psi^B_{\hat{a}} \quad \text{(no summation on } \hat{a}\text{)}$$

• 3×4 gravitino components Φ^a_A, $a = 1, 2, 3$; $A = 1, 2, 3, 4$.

Supersymmetric action (first order form)

$$S = \int dt \left[\pi_a \, \dot{\beta}^a + p_{\varphi^a} \, \dot{\varphi}^a + \frac{i}{2} \, G_{ab} \, \Phi_A^a \, \dot{\Phi}_A^b + \bar{\Psi}'^A_{\hat{0}} \, \mathcal{S}_A - \tilde{N} H - N^a H_a \right]$$

G_{ab}: Lorentzian-signature quadratic form:

$$G_{ab} \, d\beta^a \, d\beta^b \equiv \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a \right)^2$$

G_{ab} defines the kinetic terms of the gravitino, as well as those of the β^a's:

$$\frac{1}{2} \, G_{ab} \, \dot{\beta}^a \, \dot{\beta}^b$$

Lagrange multipliers $\longrightarrow$ Constraints $\mathcal{S}_A \approx 0$, $H \approx 0$, $H_a \approx 0$

Quantization

• Bosonic dof:

$$\widehat{\pi}_a = -i\,\frac{\partial}{\partial \beta^a}\ ;\quad \widehat{p}_{\varphi_a} = -i\,\frac{\partial}{\partial \varphi_a}$$

• Fermionic dof:

$$\widehat{\Phi}_A^a\,\widehat{\Phi}_B^b + \widehat{\Phi}_B^b\,\widehat{\Phi}_A^a = G^{ab}\,\delta_{AB}$$

This is the Clifford algebra $\mathrm{Spin}\,(8^+, 4^-)$

• The wave function of the universe $\Psi_\sigma(\beta^a, \varphi_a)$ is a 64-dimensional spinor of $\mathrm{Spin}\,(8,4)$ and the gravitino operators Φ_A^a are 64×64 "gamma matrices" acting on Ψ_σ, $\sigma = 1, \ldots, 64$

• Crucially depends on the terms cubic and quartic in Fermions

Dirac Quantization of the Constraints

$$\widehat{\mathcal{S}}_A \, \Psi = 0 \, , \quad \widehat{H} \, \Psi = 0 \, , \quad \widehat{H}_a \, \Psi = 0$$

Diffeomorphism constraint $\Leftrightarrow \widehat{p}_{\varphi_a} \Psi = -i \frac{\partial}{\partial \varphi_a} \Psi = 0$: "s wave" w.r.t. the Euler angles

$\longrightarrow$ Wave function $\Psi(\beta^a)$ submitted to constraints

$$\widehat{\mathcal{S}}_A(\widehat{\pi}, \beta, \widehat{\Phi}) \, \Psi(\beta) = 0 \, , \quad \widehat{H}(\widehat{\pi}, \beta, \widehat{\Phi}) \, \Psi(\beta) = 0$$

$\widehat{\pi}_a = -i \frac{\partial}{\partial \beta^a} \Rightarrow 4 \times 64 + 64$ PDE's for the 64 functions $\Psi_\sigma(\beta^1, \beta^2, \beta^3)$

Heavily overdetermined system of PDE's

Explicit form of the SUSY constraints ($\gamma^5 \equiv \gamma^{\hat{0}\hat{1}\hat{2}\hat{3}}$, $\beta_{12} \equiv \beta^1 - \beta^2$, $\widehat{\Phi}^{12} \equiv \widehat{\Phi}^1 - \widehat{\Phi}^2$)

$$\begin{aligned}
\widehat{\mathcal{S}}_A &= -\frac{1}{2}\sum_a \widehat{\pi}_a\,\Phi^a_A + \frac{1}{2}\sum_a e^{-2\beta^a}(\gamma^5\,\Phi^a)_A \\
&- \frac{1}{8}\coth\beta_{12}(\widehat{S}_{12}(\gamma^{12}\,\widehat{\Phi}^{12})_A + (\gamma^{12}\,\widehat{\Phi}^{12})_A\,\widehat{S}_{12}) \\
&+ \text{cyclic}_{(123)} + \frac{1}{2}(\widehat{\mathcal{S}}^{\text{cubic}}_A + \widehat{\mathcal{S}}^{\text{cubic}\,\dagger}_A)
\end{aligned}$$

$$\begin{aligned}
\widehat{S}_{12}(\widehat{\Phi}) &= \frac{1}{2}[(\bar{\widehat{\Phi}}^3\,\gamma^{\hat{0}\hat{1}\hat{2}}(\widehat{\Phi}^1 + \widehat{\Phi}^2)) + (\bar{\widehat{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^1) \\
&+ (\bar{\widehat{\Phi}}^2\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2) - (\bar{\widehat{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2)]\,,
\end{aligned}$$

$$\begin{aligned}
\widehat{\mathcal{S}}^{\text{cubic}}_A &= \frac{1}{4}\sum_a(\bar{\widehat{\Psi}}_0, \gamma^{\hat{0}}\,\widehat{\Psi}_{\hat{a}})\,\gamma^{\hat{0}}\,\widehat{\Psi}^A_{\hat{a}} - \frac{1}{8}\sum_{a,b}(\bar{\widehat{\Psi}}_{\hat{a}}\,\gamma^{\hat{0}}\,\widehat{\Psi}_{\hat{b}})\,\gamma^{\hat{a}}\,\widehat{\Psi}^A_{\hat{b}} \\
&+ \frac{1}{8}\sum_{a,b}(\bar{\widehat{\Psi}}_0, \gamma^{\hat{a}}\,\Psi_{\hat{b}})(\gamma^{\hat{a}}\,\Psi^A_{\hat{b}} + \gamma^{\hat{b}}\,\Psi^A_{\hat{a}})\,,
\end{aligned}$$

(Open) Superalgebra satisfied by the $\widehat{\mathcal{S}}_A$'s and $\widehat{H}$

$$\widehat{\mathcal{S}}_A \, \widehat{\mathcal{S}}_B + \widehat{\mathcal{S}}_B \, \widehat{\mathcal{S}}_A = 4 \, i \sum_C \widehat{L}_{AB}^C (\beta) \, \widehat{\mathcal{S}}_C + \frac{1}{2} \, \widehat{H} \, \delta_{AB}$$

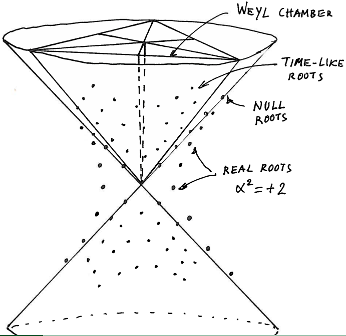
$$[\widehat{\mathcal{S}}_A , \widehat{H}] = \widehat{M}_A^B \, \widehat{\mathcal{S}}_B + \widehat{N}_A \widehat{H}$$

Root diagram of $AE_3 = A_1^{++}$

3-dimensional Lorentzian-signature space: metric in Cartan sub-algebra

$$G_{ab}\, d\beta^a\, d\beta^b = \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a\right)^2$$

(directly linked to Einstein action: $K_{ij}^2 - (K_i^i)^2$)



Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$2\,\widehat{H} = G^{ab}(\widehat{\pi}_a + i\,A_a)(\widehat{\pi}_b + i\,A_b) + \widehat{\mu}^2 + W_g^{\rm bos}(\beta) + \widehat{W}_g^{\rm spin}(\beta) + \widehat{W}_{\rm sym}^{\rm spin}(\beta)\,.$$

$G_{ab} \leftrightarrow$ metric in Cartan subalgebra of AE_3

$$W_g^{\rm bos}(\beta) = \frac{1}{2}\,e^{-2\alpha_{11}^g(\beta)} - e^{-2\alpha_{23}^g(\beta)} + {\rm cyclic}_{123}$$

$$\widehat{W}_g^{\rm spin}(\beta,\widehat{\Phi}) = e^{-\alpha_{11}^g(\beta)}\widehat{J}_{11}(\widehat{\Phi}) + e^{-\alpha_{22}^g(\beta)}\widehat{J}_{22}(\widehat{\Phi}) + e^{-\alpha_{33}^g(\beta)}\widehat{J}_{33}(\widehat{\Phi})\,.$$

Linear forms $\alpha_{ab}^g(\beta) = \beta^a + \beta^b \Leftrightarrow$ six level-1 roots of AE_3

$$\widehat{W}_{\rm sym}^{\rm spin}(\beta) = \frac{1}{2}\,\frac{(\widehat{S}_{12}(\widehat{\Phi}))^2 - 1}{\sinh^2 \alpha_{12}^{\rm sym}(\beta)} + {\rm cyclic}_{123}\,,$$

Linear forms $\alpha_{12}^{\rm sym}(\beta) = \beta^1 - \beta^2$, $\alpha_{23}^{\rm sym}(\beta) = \beta^2 - \beta^3$, $\alpha_{31}^{\rm sym}(\beta) = \beta^3 - \beta^1$
$\Leftrightarrow$ three level-0 roots of AE_3

Spin dependent (Clifford) Operators coupled to AE_3 roots

$$\widehat{S}_{12}(\widehat{\Phi}) = \frac{1}{2}[(\bar{\widehat{\Phi}}^3 \gamma^{\hat{0}\hat{1}\hat{2}}(\widehat{\Phi}^1 + \widehat{\Phi}^2)) + (\bar{\widehat{\Phi}}^1 \gamma^{\hat{0}\hat{1}\hat{2}} \widehat{\Phi}^1) + (\bar{\widehat{\Phi}}^2 \gamma^{\hat{0}\hat{1}\hat{2}} \widehat{\Phi}^2) - (\bar{\widehat{\Phi}}^1 \gamma^{\hat{0}\hat{1}\hat{2}} \widehat{\Phi}^2)],$$

$$\widehat{J}_{11}(\widehat{\Phi}) = \frac{1}{2}[\ \bar{\widehat{\Phi}}^1 \gamma^{\hat{1}\hat{2}\hat{3}}(4\widehat{\Phi}^1 + \widehat{\Phi}^2 + \widehat{\Phi}^3) + \bar{\widehat{\Phi}}^2 \gamma^{\hat{1}\hat{2}\hat{3}} \widehat{\Phi}^3].$$

- $\widehat{S}_{12}$, $\widehat{S}_{23}$, $\widehat{S}_{31}$, $\widehat{J}_{11}$, $\widehat{J}_{22}$, $\widehat{J}_{33}$ generate (via commutators) a 64-dimensional representation of the (infinite-dimensional) "maximally compact" sub-algebra $K(AE_3) \subset AE_3$. [The fixed set of the (linear) Chevalley involution, $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(h_i) = -h_i$, which is generated by $x_i = e_i - f_i$.]

The "squared-mass" Quartic Operator $\widehat{\mu}^2$ in $\widehat{H}$

In the middle of the Weyl chamber (far from all the hyperplanes $\alpha_i(\beta) = 0$):

$$2\,\widehat{H} \simeq \widehat{\pi}^2 + \widehat{\mu}^2$$

where $\widehat{\mu}^2 \sim \sum \widehat{\Phi}^4$ gathers many complicated quartic-in-fermions terms (including $\sum \widehat{S}^2_{ab}$ and the infamous ψ^4 terms of supergravity).

Remarkable Kac-Moody-related facts:

- $\widehat{\mu}^2$ *commutes* with the $K(AE_3)$ generators $\widehat{S}_{ab}$,$\widehat{J}_{ab}$
- $\widehat{\mu}^2$ is $\sim$ the square of a very simple operator $\in$ Center

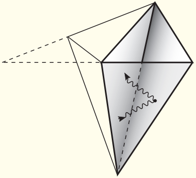
$$\widehat{\mu}^2 = \frac{1}{2} - \frac{7}{8}\,\widehat{C}_F^2$$

where $\widehat{C}_F := \frac{1}{2}\,G_{ab}\,\widehat{\overline{\Phi}^a}\,\gamma^{\hat{1}\hat{2}\hat{3}}\,\widehat{\Phi}^b$.

Solutions of SUSY constraints

Overdetermined system of four 64-component Dirac-like equations

$$\widehat{\mathcal{S}}_A \Psi = \left(\frac{i}{2} \Phi^a_A \frac{\partial}{\partial \beta^a} + U(\beta) \Phi\Phi\Phi \right) \Psi = 0$$



Space of solutions is a mixture of "discrete-spectrum states" and "continuous-spectrum states", depending on fermion number $N_F = C_F - 3$. $\exists$ solutions for both even and odd N_F.

$\exists$ continuous-spectrum states (parametrized by initial data comprising arbitrary *functions*) at $C_F = -1, 0, +1$.

Completion of inconclusive studies started long ago: D'Eath 93, D'Eath-Hawking-Obregon 93, Csordas-Graham 95, Obregon 98, ...

Solution space of quantum susy Bianchi IX: $N_F = 0$

Level $N_F = 0$: $\exists$ unique "ground state" $|f\rangle = C\, f_0(\beta)\, |0\rangle_-$ with

$$f_0(\beta) = abc\left[(b^2 - a^2)(c^2 - b^2)(c^2 - a^2)\right]^{3/8} e^{-\frac{1}{2}(a^2+b^2+c^2)} |\, 0\,\rangle_-$$

This "ground state" is localized in the middle of β space (or of a Weyl chamber) and decays in all directions in β space: small volume, large volume, large anisotropies. It describes a quantum universe which oscillates in shape and size, but stays of Planckian size

$\exists$ similar "discrete-spectrum" states at $N_F = 1, 2, 4, 5, 6$; however, it is only at levels $N_F = 0$ and 1 that these states decay in all directions and are square integrable at the symmetry walls.

Continuous-spectrum solutions at $N_F = 2, 3, 4$: Quantum Supersymmetric Billiard

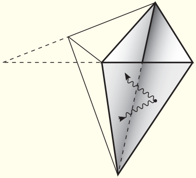
At $N_F = 2$, $\exists$ solutions of the type $k_{(ab)}(\beta)\,\tilde{b}^a_+\,\tilde{b}^b_-\,|0\rangle_-$ with a symmetric $k_{ab}(\beta) = f^{+-}_{(ab)}(\beta)$, with 6 components, that satisfies Maxwell-type equations in β space similar to $\delta k \sim 0$, $dk \sim 0$.

The spinorial wave function of the universe $\Psi(\beta^a)$ propagates within the (various) Weyl chamber(s) and "reflects" on the walls (= simple roots of AE_3). In the small-wavelength limit, the "reflection operators" define a spinorial extension of the Weyl group of AE_3 (Damour Hillmann 09) defined within some subspaces of $\mathrm{Spin}(8,4)$

$$\widehat{\mathcal{R}}_{\alpha_i} = \exp\left(-i\,\frac{\pi}{2}\,\widehat{\varepsilon}_{\alpha_i}\,\widehat{J}_{\alpha_i}\right)$$

with

$$\widehat{J}_{\alpha_i} = \{\widehat{S}_{23}, \widehat{S}_{31}, \widehat{J}_{11}\} \text{ and } \widehat{\varepsilon}^{\,2}_{\alpha_i} = \mathrm{Id}$$



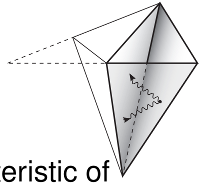
Hidden Kac-Moody structure of the spinor reflection operators

Dynamically computed reflection operators:

$$\mathcal{R}_{\alpha_{23}}^{\mathbf{6},N_F=2,WKB} = e^{-\frac{i\pi}{2}}\, e^{\pm\frac{i\pi}{2}|\widehat{S}_{23}|_{\mathbf{6},N_F=2}}\,,$$

$$\mathcal{R}_{\alpha_{12}}^{\mathbf{6},N_F=2,WKB} = e^{-\frac{i\pi}{2}}\, e^{\pm\frac{i\pi}{2}|\widehat{S}_{12}|_{\mathbf{6},N_F=2}}\,,$$

$$\mathcal{R}_{\alpha_{11}}^{\mathbf{6},N_F=2} = e^{-i\frac{\pi}{2}}\, e^{-i\frac{\pi}{2}\widehat{J}_{11}{}^{\mathbf{6},N_F=2}}\,,$$



The $\mathcal{R}_{\alpha_i}$'s satisfy generalized Coxeter relations characteristic of spin-extended Weyl groups (Ghatei, Horn, Köhl, Weiss, 2016)

$$r_i^8 = 1;$$

$$r_i r_j r_i \cdots = r_j r_i r_j \cdots \text{ “braid" relations with } m_{ij} \text{ factors on each side}\,.$$

$$r_j^{-1} r_i^2 r_j = r_i^2 r_j^{2n_{ij}}\,,$$

with some (precisely defined) integers m_{ij} and n_{ij}.

Fermions and their dominance near the singularity

Crucial issue of boundary condition near a big bang or big crunch or black hole singularity: DeWitt 67, Vilenkin 82 ..., Hartle-Hawking 83 ..., ..., Horowitz-Maldacena 03

Finding in Bianchi IX SUGRA: the WDW squared-mass term $\widehat{\mu}^2$ is *negative* (i.e. tachyonic) in most of the Hilbert space (44 among 64).

$$\mu^2 = \left(-\frac{59}{8}\Big|_0^1, -3\Big|_1^6, -\frac{3}{8}\Big|_2^{15}, +\frac{1}{2}\Big|_3^{20}, -\frac{3}{8}\Big|_4^{15}, -3\Big|_5^6, -\frac{59}{8}\Big|_6^1 \right) \quad (1)$$

This is a quantum effect quartic in fermions:

$$\rho_4 \sim \psi^4 \sim \mu^2 \left(\mathcal{V}_3\right)^{-2} = \mu^2 \left(a\,b\,c\right)^{-2} = \mu^2 \bar{a}^{-6}$$

which dominates the other contributions near a small volume singularity

Bouncing Universes and Quantum Boundary Conditions at a Spacelike Singularity ?

A "stiff", $\rho_4 = p_4$, negative $\rho_4 < 0$ contribution classically leads to an avoidance of a singularity, i.e. a bounce of the universe. Quantum mechanically, the general solution of the WDW equation (in "hyperbolic polar coordinates" $\beta^a = \rho\gamma^a$)

$$\left(\frac{1}{\rho}\partial_\rho^2\,\rho - \frac{1}{\rho^2}\Delta_\gamma + \hat{\mu}^2\right)\Psi'(\rho,\gamma^a) = 0$$

behaves, after a quantum-billiard mode-expansion $\Psi'(\rho,\gamma^a) = \sum_n R_n(\rho)\, Y_n(\gamma^a)$, as

$$\rho\, R_n(\rho) \equiv u_n(\rho) \approx a_n e^{-|\mu|\rho} + b_n e^{+|\mu|\rho}\ ,\ \text{as } \rho \to +\infty$$

This suggests to impose the boundary condition $\Psi' \sim e^{-|\mu|\rho} \to 0$ at the singularity, which is, for a black hole crunch, a type of "final-state" boundary condition (à la Horowitz-Maldacena), which would represent a quantum avoidance of the singularity ?

Conclusions

- The BKL approach to the description of general spacelike cosmological singularities has been confirmed by many analytical and numerical studies, though a mathematical proof is still lacking (except for some nonchaotic cases).
- The BKL-type cosmological billiard dynamics is equivalent to billiard motion in the Weyl chamber of an hyperbolic Kac-Moody algebra (AE_3 for GR_4, E_{10} for SUGRA_{11}).
- The evidence for a hidden Kac-Moody structure goes much beyond the billiard limit (both in bosonic and fermionic EOM and in classical/quantum effects). It suggests a gravity/coset correspondence: gravity dynamics $\leftrightarrow$ massless particle on infinite-dimensional (Lorentzian-signature) Kac-Moody coset G/K.
- The case study of the quantum dynamics of a triaxially squashed 3-sphere (Bianchi IX model) in (simple, $D = 4$) supergravity confirms the hidden presence of hyperbolic Kac-Moody structures (AE_3 and $K(AE_3)$) in supergravity, especially in the fermionic sector.
- Quartic terms in the gravitino might lead to a quantum avoidance of the singularity.