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Structure of coherent vortices caused by the inverse cascade of 2d turbulence and related problems

Igor Kolokolov and Vladimir Lebedev

Landau Institute

We consider the turbulence that is excited by an external pumping in a thin fluid layer. It can be treated as two-dimensional $(2d)$ on scales larger than the layer thickness. The pumping correlation length l is assumed to be much larger than the layer thickness, however, it should be much less that the box size L. The Reynolds number is assumed to be large.

The pumped $2d$ turbulence is described by the equation for vorticity ω

$$\partial_t \omega + \boldsymbol{v}\nabla\omega = \nabla \times \boldsymbol{f} + \nu\nabla^2\omega - \alpha\omega,$$

where $\boldsymbol{v}$ is velocity, $\boldsymbol{f}$ is pumping force per unit mass, ν is viscosity and α is bottom friction coefficient. The pumping force is assumed to be correlated at a scale $l \ll L$ and to be random in time.

There are two quadratic dissipationless integrals of motion, energy and enstrophy:

$$\int dx\ dy\ v^2, \qquad \int dx\ dy\ \omega^2.$$

Pumped turbulence – two cascades: enstrophy flows to small scales whereas energy flows to large scales, being dissipated by viscosity and friction, respectively (Kraichnan 1967, Leith 1968, Batchelor 1969).

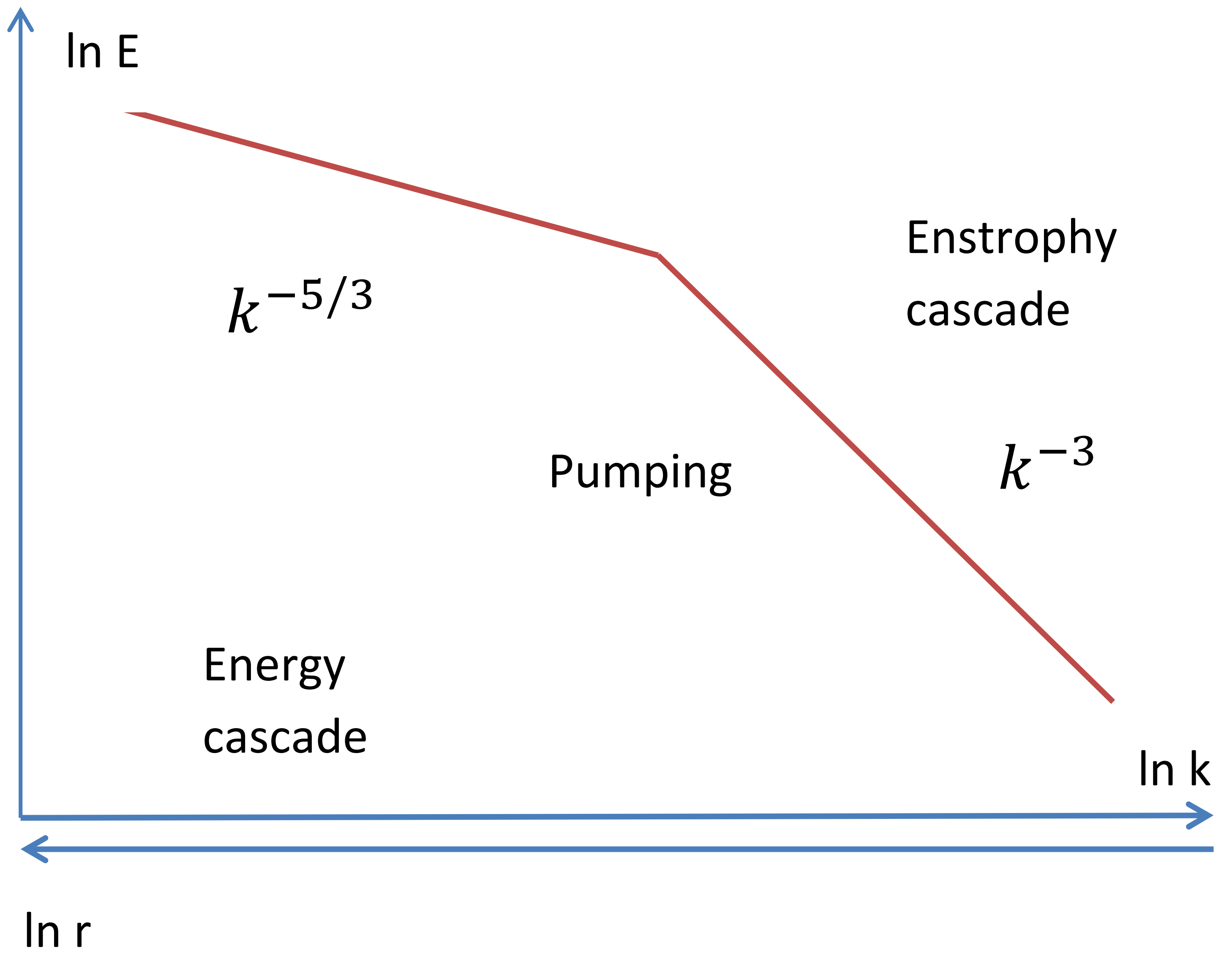
Constancy of the energy and enstrophy fluxes imply the proportionality laws

$$\langle (v_1 - v_2)\omega_1\omega_2 \rangle \propto r, \quad r \ll l;$$
$$\langle |v_1 - v_2|^3 \rangle \propto r, \quad r \gg l.$$

Suggest the normal scaling $v_1 - v_2 \propto r$ in the direct cascade and $v_1 - v_2 \propto r^{1/3}$ in the inverse cascade. The spectrum

$$\langle v_1 v_2 \rangle = \int \frac{dk}{2\pi} e^{ikr} E(k),$$

Then $E(k) \propto k^{-3}$ for the direct (enstrophy) cascade $E(k) \propto k^{-5/3}$ for the inverse (energy) cascade. Direct cascade – logarithmic correlation functions of vorticity (Falkovich, Lebedev 1994). Inverse cascade – an absence of anomalous scaling (Paret and Tabeling 1998, Boffetta, Celani and Vergassola 2000).

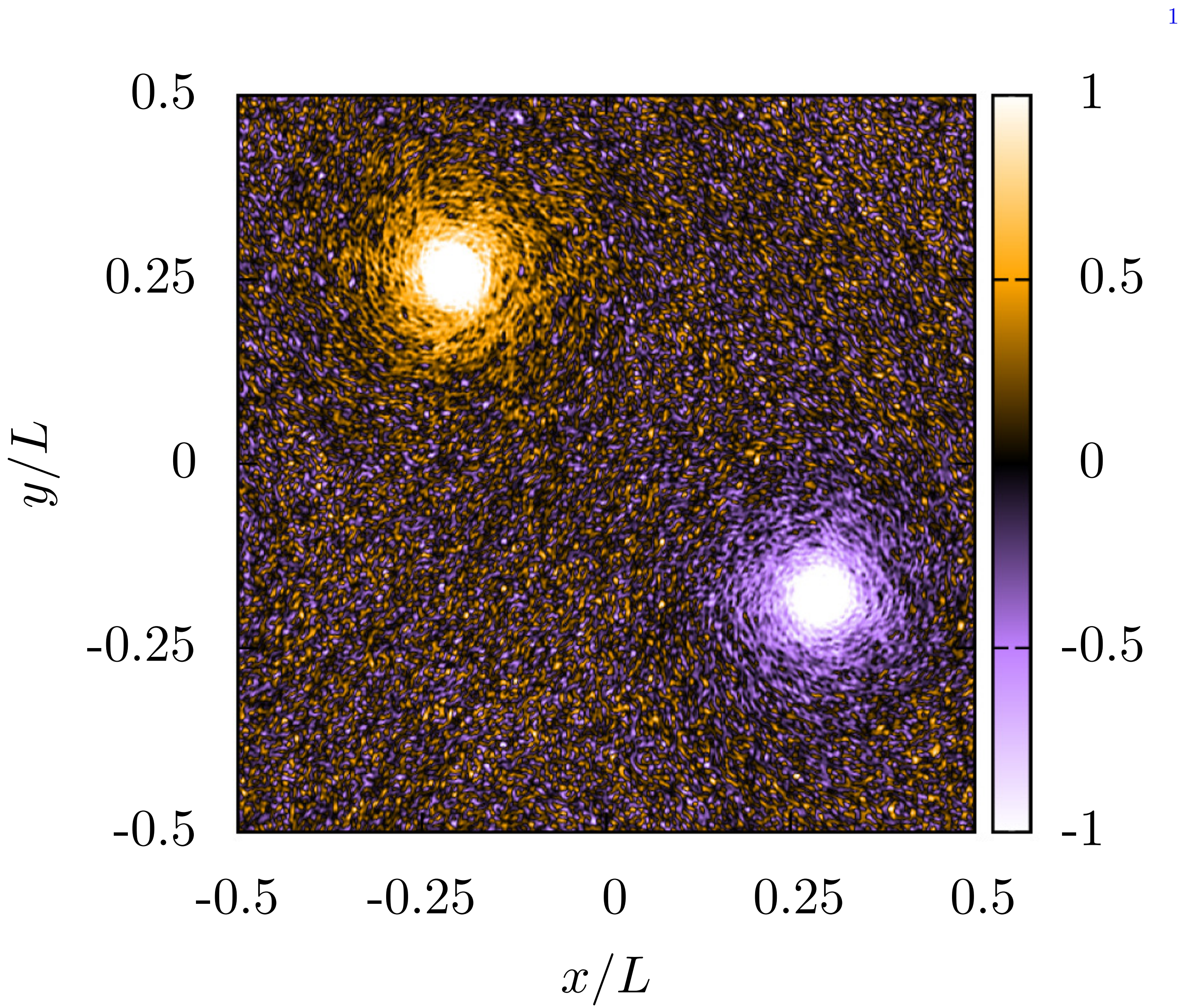

ln E
$k^{-5/3}$
Enstrophy
cascade
Pumping
k^{-3}
Energy
cascade
ln k
ln r

In an infinite system the inverse cascade is terminated by the friction at the scale $L_\alpha \sim \epsilon^{1/2}\alpha^{-3/2}$ where ϵ is the energy production rate per unit mass. If the box size $L < L_\alpha$ then the energy accumulates at L: experiment (Shats, Xia, Punzmann and Falkovich 2007) and numerics (Chertkov, Connaughton, Kolokolov and Lebedev 2007). Coherent structures are formed!

The coherent velocity profile arises at a time $t \sim t_L = L^{2/3}\epsilon^{-1/3}$. After that the major part of the pumped energy is accumulated at scales $\sim L$. Therefore typical large-scale velocity $\sim \sqrt{\epsilon t}$ increases as time grows. The stage is terminated at time $t \sim \alpha^{-1}$. After that, at $t \gg \alpha^{-1}$, some steady (statistically homogeneous in time) state is realized.

It is characterized by an average flow $\boldsymbol{V}$ and fluctuations on this background. As we saw, the average flow in the periodic setup consists of two coherent vortices. In laboratory experiments in a square box the average flow consists typically of five vortices: a big vortex in the center of the box and four counter-rotating smaller vortices in the corners of the box.

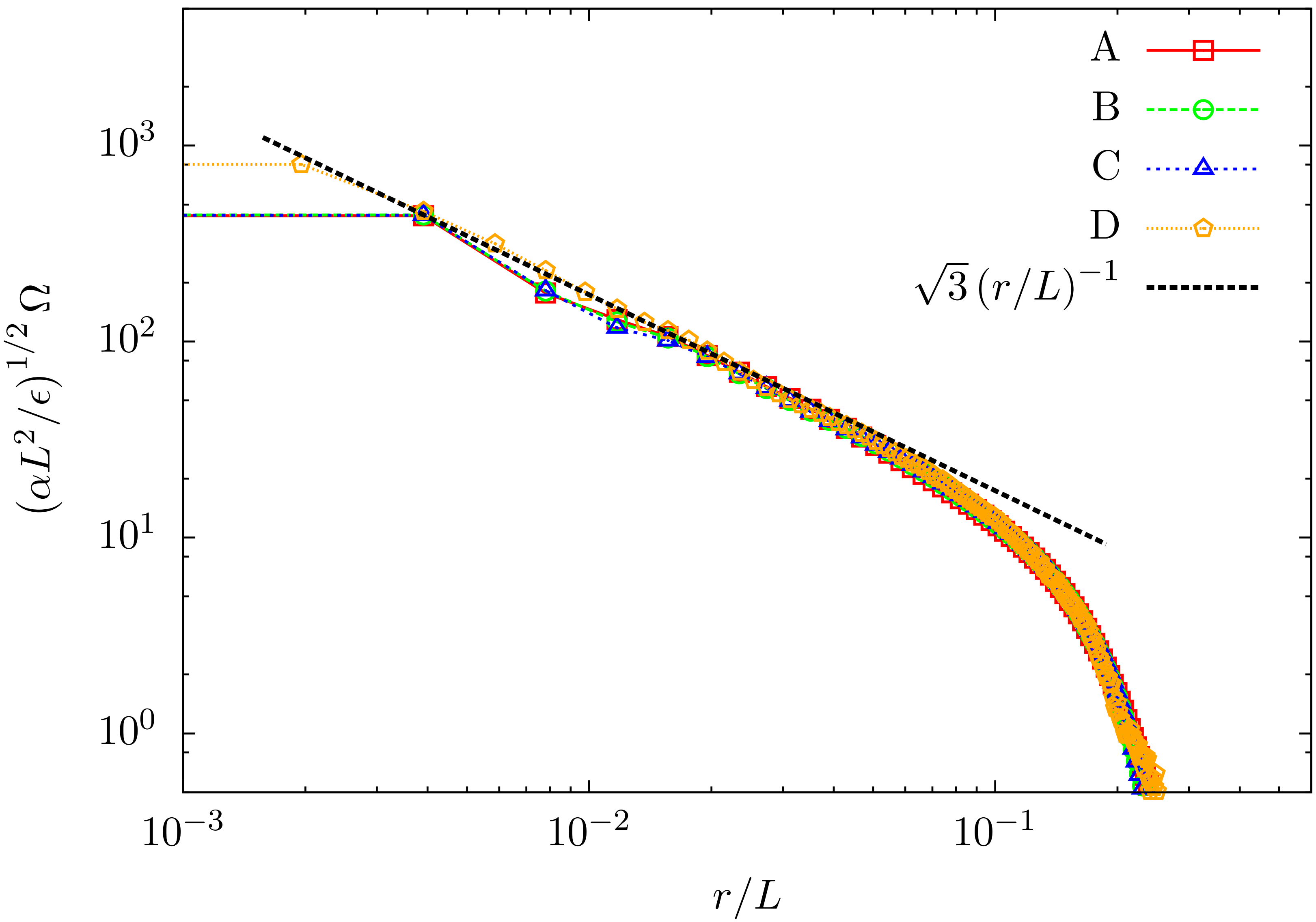
To extract properties of the average flow inside the vortex, one places origin at the vortex center. Both, experiment and numerics, show that the vortices are isotropic in average: the average polar velocity U and the average vorticity Ω are functions of the separation from the vortex center r. Snapshot of numerics Laurie, Bofetta, Falkovich, Kolokolov, Lebedev 2014.

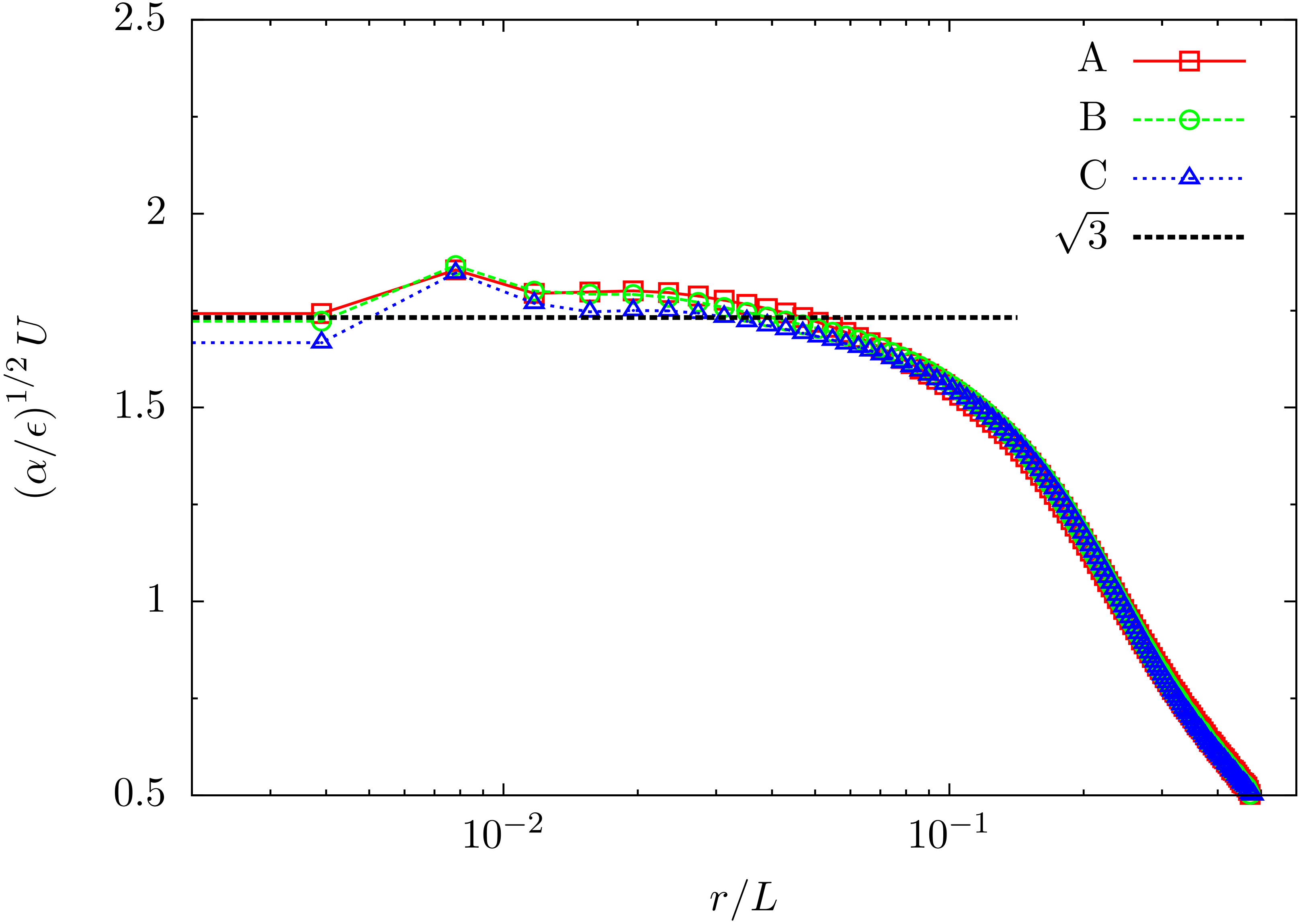


1
0.5
0.25
0
-0.25
-0.5
y/L
1
0.5
0
-0.5
-1
-0.5
-0.25
0
0.25
0.5
x/L

There is the hyperbolic region where the flow velocity v is estimated as $\sqrt{\epsilon/\alpha}$ and the flow vorticity ω is estimated as $\sqrt{\epsilon/\alpha}/L$ from the energy balance: pumping versus friction. The average Ω increases toward the vortex center. One observes power-like behavior of Ω outside a relatively small core. The core radius is determined by viscosity.

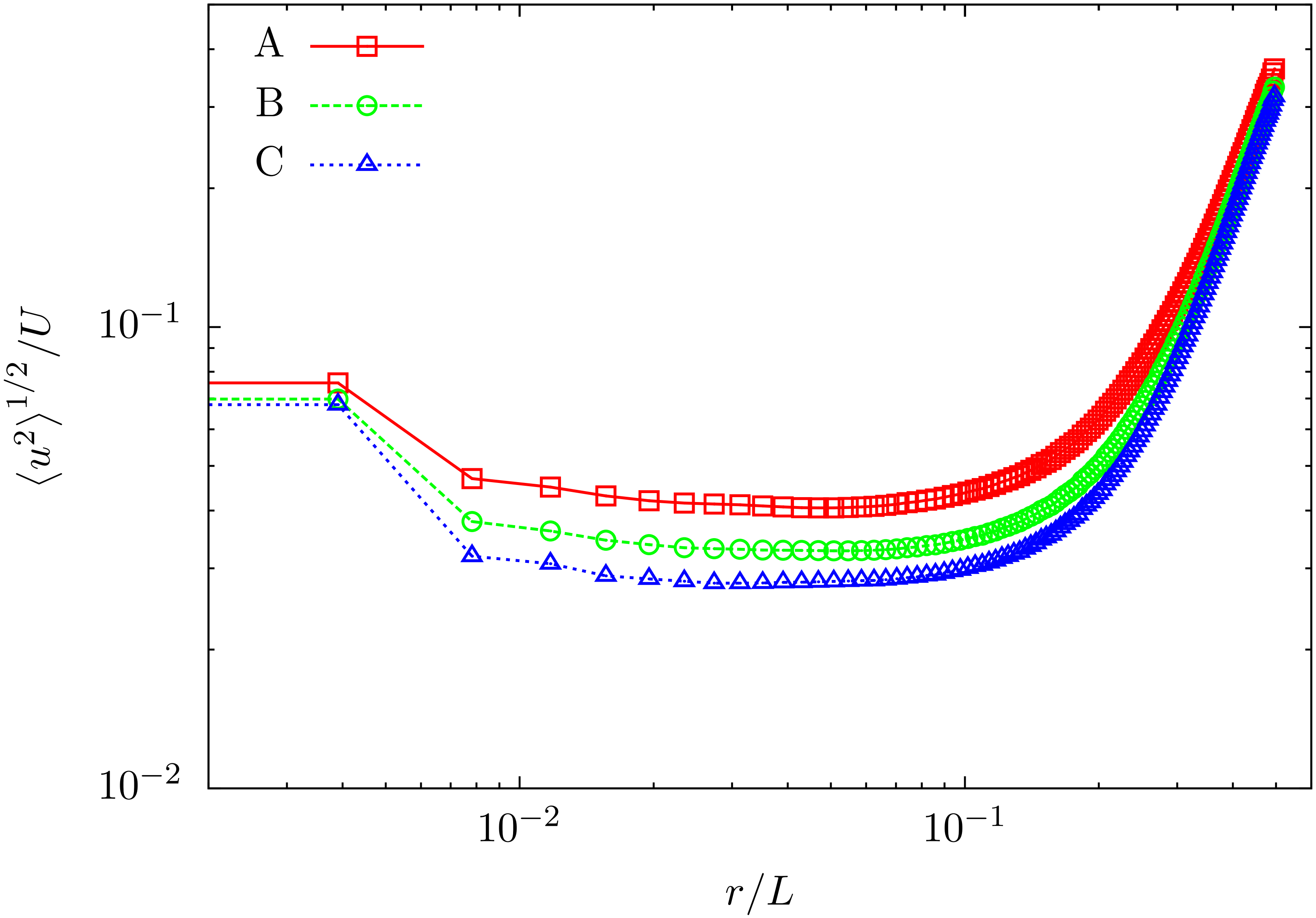
The numerics (Laurie, Bofetta, Falkovich, Kolokolov, Lebedev 2014), where pumping was short correlated in time, reveals the universal behavior $U = \mathrm{const}$ and $\Omega \propto r^{-1}$ outside the viscous core. The law is correct provided $r \ll L$ where L is the box size. Inside the core the average vorticity Ω is saturated and the average velocity U tends to zero $U \propto r$.

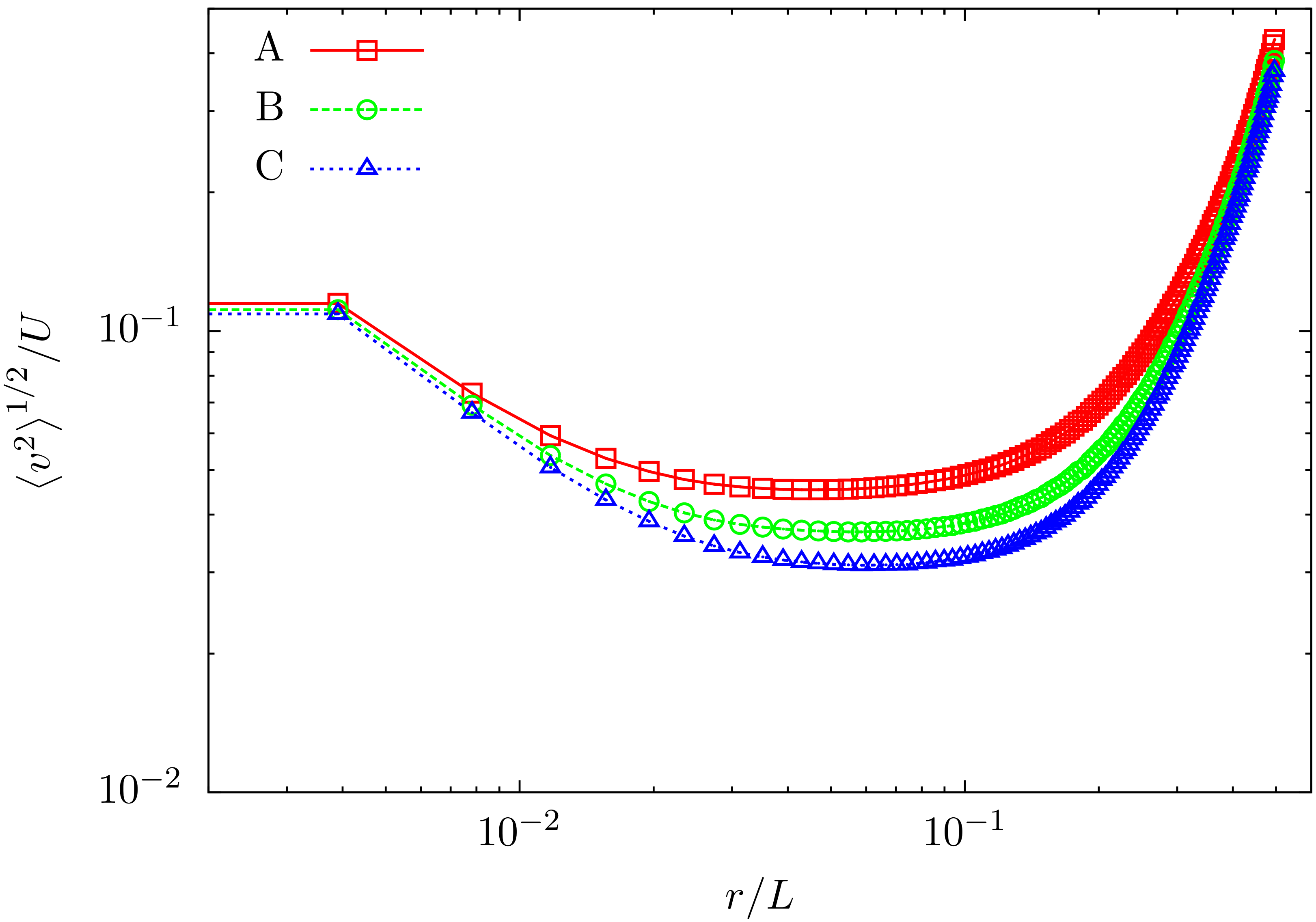

A
B
C
D
$\sqrt{3}\,(r/L)^{-1}$
10^3
10^2
10^1
10^0
10^{-3}
10^{-2}
10^{-1}
r/L
$(\alpha L^2/\epsilon)^{1/2}\,\Omega$



2.5
2
1.5
1
0.5
$(\alpha/\epsilon)^{1/2}U$
10^{-2}
10^{-1}
r/L
A
B
C
$\sqrt{3}$

In the universal region $u, v \ll U$, where u, v are polar and radial components of the fluctuating velocity. It is a consequence of the large value of the mean velocity gradient $\sim U/r$, growing toward the center of the vortex. The relative strength of fluctuations increases as r grows and on the periphery where $r \sim L$, fluctuations become stronger than inside the vortex.


A
B
C
10^{-1}
10^{-2}
$\langle u^2 \rangle^{1/2}/U$
10^{-2}
10^{-1}
r/L


A
B
C
10^{-1}
10^{-2}
$\langle v^2 \rangle^{1/2}/U$
10^{-2}
10^{-1}
r/L

The flat velocity profile was observed experimentally where the electromagnet technique was used (Orlov, Brazhnikov, Levchenko, 2018). The main trick was to switch the pumping off and to examine the profile after some period during which fluctuations are died. After that a complicated decay is observed that is a subject of future investigations.

Averaging the basic NS equation, one obtains the following equation for the vortex velocity

$$\alpha U = -\left(\partial_r + \frac{2}{r}\right)\langle uv\rangle + \nu\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)U$$

Comparing the terms in the equation one finds the core radius $R_c \sim (\nu/\alpha)^{1/2}$. To establish the flow profile outsize the core one has to find the average $\langle uv\rangle$ entering the equation for U.

In the work (Laurie, Bofetta, Falkovich, Kolokolov, Lebedev 2014) we derived the value of the average velocity in the universal region $U = \sqrt{3\epsilon/\alpha}$, where ϵ is the energy production rate per unit mass. The expression was derived from the conservation laws where some terms were neglected partly by symmetry partly by heuristic arguments. It is in a good agreement with numerics.

Further investigation (Kolokolov. Lebedev, 2016) shown that inside the universal region the flow fluctuations interact weakly and can be described in terms of the passive equation

$$\partial_t \varpi + (U/r)\partial_\varphi \varpi + v\partial_r \Omega = \phi - \hat{\Gamma}\varpi,$$

where ϖ is the fluctuating vorticity and $\hat{\Gamma}$ includes both bottom friction and viscosity. We solve the equation and then find $\langle uv \rangle$.

Direct calculations show that $\langle uv \rangle$=0 if $\widehat{\Gamma} = 0$. The property is explained by symmetry reasoning: if $\widehat{\Gamma} = 0$ then the dynamic equations are invariant under the following combined transformation (kept U untouched)

$$t \to -t,\ \varpi \to \varpi,\ \varphi \to -\varphi,\ r \to r,\ v \to -v,\ u \to u.$$

Obviously, the average $\langle uv \rangle$ changes its sign at the transformation and is zero.

Next: if we take a nonzero $\widehat{\Gamma}$ then $\langle uv \rangle$ becomes nonzero and independent of $\widehat{\Gamma}$ (a kind of dissipation anomaly):

$$\langle uv \rangle = \epsilon / \Sigma,$$

where Σ is the local shear rate $\Sigma = \partial_r U - U/r$. Note that $\langle uv \rangle$ is gained at small scales where viscosity comes into game. There dissipation rate balances Σ.

Thus we arrive at the equation

$$\alpha U = -\left(\partial_r + 2/r\right)\epsilon/\Sigma,$$

that has a solution

$$U = \sqrt{3\epsilon/\alpha}, \qquad \Sigma = -U/r.$$

The solution (flat velocity profile) is correct outside the vortex core. What is the restriction from the other side?

Our consideration was passive. The applicability condition of the approach is

$$r/U \ll l^{2/3}\epsilon^{-1/3},$$

where the right-hand side is characteristic turnover time at the pumping scale. Equating the quantities we find the radius of the universal zone $R_u \sim \epsilon^{1/6}\alpha^{-1/2}l^{2/3}$. What is outside the radius R_u?

At $r > R_u$ the situation is not passive, the direct cascade is restored and partly the inverse cascade does. Thus, nonlinearity is relevant and no consistent calculations are possible. However, one can derive an effective equation for the average velocity profile exploiting the fact that $\langle uv \rangle$ is gained at small scales and symmetry reasoning (Kolokolov. Lebedev, 2016).

Analyzing the equation, we conclude that the average vorticity Ω decreases fast (exponential in the region $r > R_u$. Therefore the average velocity can be estimated as $U \sim \sqrt{\epsilon/\alpha}\, R_u/r$, just by equating the velocity circulation to the total vorticity. Thus the average velocity in the region is smaller than fluctuations that can be estimated as $\langle v^2 \rangle \sim \epsilon/\alpha$.

One can calculate the strength of the velocity fluctuations in the universal region $\langle v^2\rangle, \langle u^2\rangle \sim k_f r\epsilon/\Sigma$ if $\Gamma k_f r \ll \Sigma$, and also $\langle v^2\rangle, \langle u^2\rangle \sim \epsilon/\Gamma$ if $\Gamma k_f r \gg \Sigma$. Any case, fluctuations are suppressed in comparison with the outer region and $\langle v^2\rangle, \langle u^2\rangle \gg \langle uv\rangle$. The zero angular harmonic should be calculated separately $\langle u_0^2\rangle \sim \epsilon/(\Gamma k_f r)$.

Since in the universal region the quasi-linear approximation is correct one can consistently calculate correlation functions of the velocity fluctuations. Small-scale correlation functions (at scales smaller than r) are calculated in Kolokolov Lebedev 2016. Correlations at scales $\sim r$ are analyzed in Frishman Herbert 2018 where analytics is confirmed by numerics.

Static pumping (Kolokolov Lebedev 2017): the force f is time-independent and has finite correlation length l. One can think about periodic case (motivated by experiment). The inequality $f \gg l\alpha^2$ is obligatory. Though all the above estimates in terms of the energy flux ϵ are correct, the value of ϵ is not fixed. One of the problems – determination of ϵ.

Possible scenario: time-independent average $\boldsymbol{V}$ with small fluctuations on its background. The scenario is self-contradictory in terms of $U(r)$. Direct calculations show that in this static case $\langle uv \rangle \propto \Gamma$, it is small. In this situation the zero angular harmonic with the scale l is much larger than U. Thus the fluctuations cannot be passive, and, probably the vortex doesn't exist.

Possible resolution: the center of the vortex fluctuates with a velocity related to the velocity fluctuations at the periphery V. Then in the reference system attached to the vortex the pumping has effectively finite correlation time, it can be estimated as $\tau = l/V$. Then the energy production rate can be estimated as $\epsilon \sim f^2\tau$.

Equating the energy production $\epsilon \sim f^2\tau$ to the energy dissipation αV^2, we arrive at the estimates

$$V \sim f^{2/3} l^{1/3} \alpha^{-1/3},$$
$$\epsilon \sim f^{4/3} l^{2/3} \alpha^{1/3},$$

for the large-scale velocity V and the energy flux ϵ. Thus the energy flux ϵ is dependent on α.

Now we should conduct calculations of $\langle uv \rangle$ for the pumping with finite correlation time and extract U from the equation

$$\alpha U + (\partial_r + 2/r)\langle uv \rangle = 0.$$

Assuming passiveness we can use the same scheme as for the short correlated case, however, the effective correlation time τ of pumping is now finite.

It is convenient to start from the equation for the small-scale vorticity ϖ that in the passive case in the reference frame attached to the vortex can be written as

$$\partial_t \varpi + (U/r)\partial_\varphi \varpi + \hat{\Gamma}\varpi = \phi,$$

where $\phi = \mathrm{curl}\, \boldsymbol{f}$, φ is the polar angle and $\hat{\Gamma}\varpi$ is the dissipation term. Next we use the shear approximation.

The difference in comparison with the short-correlated pumping is that the velocity U explicitly enters (besides Σ) the expressions for ϖ. Nevertheless, the calculations can be conducted practically to the end and we find $\langle uv\rangle \sim \epsilon/\Sigma$, the coefficient here depends on details of the statistics of V. Thus $U \sim (\epsilon/\alpha)^{1/2} \sim f^{2/3} l^{1/3} \alpha^{-1/3} \sim V$.

Though the case of static pumping essentially differs from the one of the pumping short correlated in time, the final conclusions concerning the existence of the scale interval with the flat coherent velocity profile with the value of the order of the velocity of the large-scale motion (obtained from the energy budget) is the same.

Conclusions

Coherent vortices inevitably appear as a consequence of the inverse cascade in restricted systems.

Fluctuations are strongly suppressed inside the coherent vortices while fluctuations are stronger that the average velocity profile outside the vortices.

The average velocity profile of the coherent vortices is flat, independent of the pumping details.