"Moscow zero", and solvable deformations of 2D quantum field theories

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• **"Moscow Zero"** is a phenomenon discovered by **L.D.Landau, A.A.Abrikosov, I.M.Khalatnikov** (1954) concerning the UV properties of QED. The observed value of the coupling $\alpha_{\rm ph} = 1/137$ relates to its "bare" value α_0 as

$$\frac{1}{\alpha_{\rm ph}} = \frac{1}{\alpha_0} + \beta_2 \log\frac{\Lambda}{M}, \qquad \frac{1}{\alpha_0} = \frac{1}{\alpha_{\rm ph}} - \beta_2 \log\frac{\Lambda}{M},$$

where $\beta_2 = N_f/6\pi^2$, and Λ is UV cutoff energy. When Λ increases, the bare coupling α_0 must be increased, too, in order to keep $\alpha_{\rm ph}$ fixed, equal to 1/137. And, as the second form of the equation makes explicit, the cutoff Λ has an upper bound (or "Landau-Abrikosov-Khalatnikov (LAK) scale")

$$\Lambda_* = M\,\mathrm{e}^{\frac{1}{\beta_2\,\alpha_{\rm ph}}} \simeq 10^{280}$$

at which α_0 diverges, i.e. $1/\alpha_0$ turns to zero, and then becomes negative. Since α_0 enters the action as

$$\mathcal{A}_{QCD} \sim \frac{1}{\alpha_0}\int F_{\mu\nu}^2 + ...$$

one is advised to keep Λ below Λ_*, since otherwise the path integral behaves badly for short range field fluctuations. One says that "QED is not UV complete".

Certainly, this conclusion was based on a number of admissions. The above equation relating α_0 to α_{ph} is derived if one neglects all terms in the beta function

$$\frac{d\alpha}{d\log\mu} = \beta(\alpha) = \beta_2\,\alpha^2 + ...$$

beyond the lowest order. When α grows, higher terms may become important. The higher order terms in $\mathcal{A}_{QCD}$ may become important as well.

Nonetheless, the described behavior is believed to be qualitatively correct in QCD, and indeed the phenomenon likely extends to a large class of QFT. Importantly, it agrees well with general pattern suggested by Wilson's Renormalization Group.

In general setting, Wilson's RG transformations act in

$$\Sigma \ = \ \text{the Space of Field Theories}$$

which may be regarded as the set of <u>quasilocal</u> actions

$$\mathcal{A} = \int_x \mathcal{L}\left(\phi(x), \partial_\mu\phi(x), \partial_\mu\partial_\nu\phi(x), ...\right)$$

where $\phi(x)$ stands for a collection of "fundamental fields" - the integration variables in the path integral. The path integral is assumed to have UV cutoff Λ. The action may include higher derivatives, but it is assumed that the derivative expansion converges for $|k| < \Lambda$. (At this point the question of unitarity is ignored). The quasilocal action $\mathcal{A}$ is accepted as a member of Σ provided the (Euclidean) path integrals

$$\int D[\phi]\ (...)\ e^{-\mathcal{A}[\phi]}$$

are convergent.

RG transformations represent scale transformations in field theory. If one simply rescales $x \to x/L$ with some $L > 1$, that would also change the cutoff $\Lambda \to L\,\Lambda$, that is one also needs to integrate out degrees of freedom with $|k|$ between Λ and $L\,\Lambda$. This leads to L-dependent transformation of the action $\mathcal{A} \to R_L\{\mathcal{A}\}$, or, in infinitesimal form

$$\frac{d}{dl}\mathcal{A} = B\{\mathcal{A}\}\,, \qquad l = \log L$$

Assuming that Σ may be coordinatized with an (generally infinite) set of parameters $\{\alpha^i\}$, this translates to a system of ordinary differential equations

$$\frac{d\alpha^i}{dl} = B^i(\{\alpha\}) \qquad (B^i = -\beta^i)$$

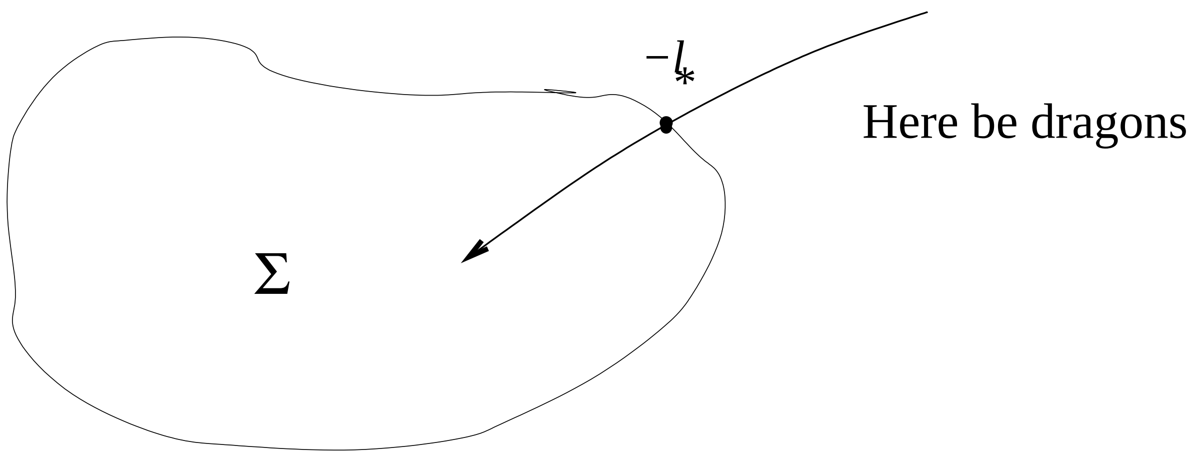
The "RG trajectories" - the integral curves of this system of differential equations - give insight into the scale dependence of physics. Large scale properties are obtained by integrating the RG equations <u>forward</u> in "RG time" l (= log of the length scale).

One does not expect any pathologies $\mathcal{A}_l$ with $l > 0$, as long as $\mathcal{A}_0$ is well defined. Indeed, it is assumed that path integral with $\mathcal{A}_0$ is convergent and well defined as QFT. Integrating out part of variables is not expected to change that.

However, as the system of differential equations, the RG flow equation can be integrated "backward", to negative values of the "RG time" l, as well.

Remark: This would be true if the number of couplings α^i was finite. In exact RG this number - the dimensionality of Σ - is infinite. There is an interesting question if the common properties of finite-dimensional systems (like uniqueness of solution) remain generally valid in exact, infinite dimensional, equations. It is usually assumed to be the case - after all, in any practical implementation of RG transformation some finite-dimensional approximation is used. But it may be one of "dangerous" assumptions. Generally, it is not completely clear how such "backward" integration agrees with generally "irreversible" nature of the RG transformations.

Although it is likely possible to integrate "backward", there are no reasons to assume that this can be done indefinitely, while staying within the space Σ along the way. In fact, one expects something opposite. Indeed, consider $\mathcal{A}_{-l}$ with $l >> 1$. Should $\mathcal{A}_{-l} \in \Sigma$, then $\mathcal{A}_0$ with cutoff Λ could be obtained from $\mathcal{A}_{-l}$ by an RG transformation, which means it would be essentially equivalent to another theory with much larger cutoff $e^l \Lambda$, i.e. with much shorter interaction range then $\mathcal{A}_0$ itself has. This is generally unlikely, which is to say that generally $\mathcal{A}_{-l}$ leaves Σ for sufficiently large l.



This picture assumes that Σ has a boundary separating "well defined" actions from the wild expanse beyond, where all "pathological" $\mathcal{A}$ lie.

Thus, given $\mathcal{A}_0$, one generally expects that at some $l = l_*$ the theory $\mathcal{A}_{-l}$ crosses that boundary, and then leaves Σ. If this happens at finite l_*, we say that the theory is "UV incomplete" (as a quantum field theory)

There is, of course, small but important subspace $\Sigma(\infty) \subset \Sigma$ for which $l_* = \infty$, i.e. the RG flow can be integrated "backwards" without limit (e.g. the flows which stem from UV fixed points, but more complicated scenarios are conceivable). This is the subspace of UV complete QFT, in which UV cutoff can be consistently removed. Non-abelian gauge theories in 4D, 3D Wilson-Fisher theory, are well known examples.

Note that, given arbitrary $\mathcal{A}_0 \in \Sigma$, the corresponding l_* is determined solely by $\mathcal{A}_0$ itself. Thus, if $l_* < \infty$, the theory $\mathcal{A}_0$ has an intrinsic UV scale

$$\Lambda_* = M\, e^{l_*}$$

where M is some "physical" mass scale, say the inverse correlation length R_c^{-1}. Thus, the "intrinsic" cutoff Λ_* is independent of the "auxiliary" RG cutoff Λ (more precisely, the ratio Λ_*/M is RG invariant).

If $l_* < \infty$, the "Landau-Abrikosov-Khalatnikov scale" Λ_* sets the upper limit for the cutoff Λ. If one assumes $\Lambda > \Lambda_*$, some kind of pathology is expected at the distances $< \Lambda_*^{-1}$. Generally, physical characteristics of the cutoff theory $\mathcal{A}_l$ are expected to develop some singularity at $l = -l_*$ (in QED $\alpha_{\text{ph}}(\Lambda)$ develops famous pole at the Landau-Abrikosov-Khalatnikov scale).

What kind of "pathology" is expected? By definition, the "theories" outside Σ can not be described by convergent path integral with a quasilocal action $\mathcal{A}[\phi]$. So, what can happen when one continues (analytically) to the scales beyond the LAK scale?

This may look like a question with no physical significance - typically, we are interested in the large scale behavior of microscopically defined systems. And the bulk of the "dragon infested area" outside Σ is likely filled with various microscopic stuff, with built in scales Λ_*.

However, the question may be of interest, from two points of view.

• One is related to attempts to understand the geometry of the "theory space" Σ. From this point of view, it is natural to associate the tangent space $T\Sigma\mid_{QFT}$ with (some simple subspace of) the space of local fields $\mathcal{F}_{QFT}$ of the given QFT. And $\mathcal{F}_{QFT}$ is furnished by composite fields $\mathcal{O}_i(\phi(x), \partial\phi(x), ...)$ with arbitrary high derivatives (one needs to include all of them if the OPE structure is to be preserved). This allows for "actions" which are not bounded from below, and/or not quasilocal. Thus, it seems understanding the "theory space" in geometric terms requires including at least some parts of the area outside Σ.

• Another perspective comes from the S-matrix theory. There is a large and growing body of evidence that if one takes a generic S-matrix, which satisfies all the standard conditions (unitarity, analyticity, crossing, bootstrap,...), there may be no "local structure" (understood as a set of operators with the conventional local commutativity condition) behind it, and that this is a norm rather then exception. Well known examples come from string theories, and this situation is expected to be typical in any theory involving quantum gravity. However, it is possible (and likely) that the class of relativistic S-matrices with no local structures is much wider than that.

In this talk I will describe an example (and indeed a class of examples) in $D = 2$, which admits some sort of "exact solution" - the so-called **"$(T\bar{T})$ flow"**. The setup is as follows.

Consider generic 2D $QFT \in \Sigma$. I will discuss in flat 2D Euclidean Space, and use notation z for its points. Complex coordinates $(\mathsf{z}, \bar{\mathsf{z}})$

$$z = (\mathsf{z}, \bar{\mathsf{z}}) : \qquad \mathsf{z} = \mathsf{x} + i\mathsf{y}\,, \qquad \bar{\mathsf{z}} = \mathsf{x} - i\mathsf{y}$$

will usually be used. I will write local fields as $\mathcal{O}(z) = \mathcal{O}(\mathsf{z}, \bar{\mathsf{z}})$.

The QFT conserves energy and momentum, and the associated local densities constitute the Energy-Momentum Tensor $T_{\mu\nu}$. We assume

$$T_{\mu\nu} = T_{\nu\mu}\,, \qquad \partial_\mu T^{\mu\nu} = 0\,.$$

Below I use CFT-inspired notations

$$T = -(2\pi)\, T_{\mathsf{zz}}\,, \qquad \bar{T} = -(2\pi)\, \bar{T}_{\bar{\mathsf{z}}\bar{\mathsf{z}}}\,, \quad \text{and} \quad \Theta = (2\pi)\, T_{\mathsf{z}\bar{\mathsf{z}}}\,,$$

in which the continuity equation takes the form

$$\partial_{\bar{\mathsf{z}}} T = \partial_{\mathsf{z}} \Theta \,, \qquad \partial_{\mathsf{z}} \bar{T} = \partial_{\bar{\mathsf{z}}} \Theta \,.$$

Some properties of the operator products of these fields follow from the above equations alone. The one important to me now is

$$T(z)\bar{T}(z') - \Theta(z)\Theta(z') = \; X(z') + \text{derivatives}\,,$$

where

$$\text{''derivatives''} = \sum_i C^{i,\mu}(z-z')\,\partial_\mu \mathcal{O}_i(z')\,.$$

The first, non-derivative term comes with the OPE coefficient $C^X(z-z') = 1$. This follows from the identity

$$\partial_{\bar{\mathsf{z}}}\Big(T(z)\bar{T}(z') - \Theta(z)\Theta(z')\Big) = \\ (\partial_{\mathsf{z}} + \partial_{\mathsf{z}'})\Big(\Theta(z)\bar{T}(z')\Big) - (\partial_{\bar{\mathsf{z}}} + \partial_{\bar{\mathsf{z}}'})\Big(\Theta(z)\Theta(z')\Big)\,,$$

and similar identity for $\partial_{\mathsf{z}}(...)$. (Use OPE for the products in the r.h.s.)

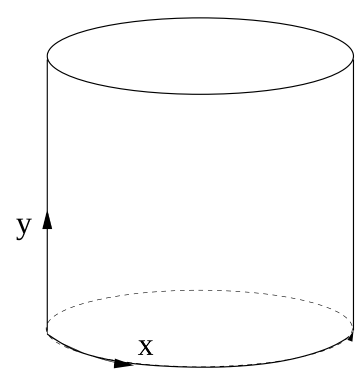
This uniquely defines a scalar local field, of exact dimension 4,

$$X(z) \equiv (T\bar{T})(z)\,,$$

uniquely, up to adding derivatives. Since scalar fields (modulo derivatives) are vectors in the tangent space $T\Sigma|_{QFT}$, the field X defines uniquely a tangent vector

$$X \in T\Sigma|_{QFT}\,.$$

A number of consequences follow. For instance, consider given QFT in the geometry of a cylinder, with the spatial coordinate compactified on a circle, $\mathsf{x} \sim \mathsf{x} + R$,



At finite size the Hamiltonian has discrete spectrum - I assume that the QFT is compact, for simplicity, and I denote $|n\rangle$ the corresponding eigenstates. Take the diagonal matrix element $\langle n \mid ... \mid n\rangle$ of the above operator identity. Derivatives do not contribute, and the cluster property at large separations suggests

$$\langle n \mid (T\bar{T})(z) \mid n\rangle = \langle n \mid T(z) \mid n\rangle\langle n \mid \bar{T}(z) \mid n\rangle - \langle n \mid \Theta(z) \mid n\rangle^2 \,.$$

Moreover, the expectation values in the r.h.s. can be expressed in terms of the eigenvalues

$$E_n(R)\,, \qquad P_n(R) = \frac{2\pi\,\ell_n}{R}\,.$$

because

$$\langle\, n \mid T_{\mathrm{yy}} \mid n\,\rangle = -\frac{1}{R}\,E_n(R)\,, \qquad \langle\, n \mid T_{\mathrm{xx}} \mid n\,\rangle = -\frac{d}{dR}\,E_n(R)\,,$$

$$\langle\, n \mid T_{\mathrm{xy}} \mid n\,\rangle = \frac{i}{R}\,P_n(R)\,.$$

One derives

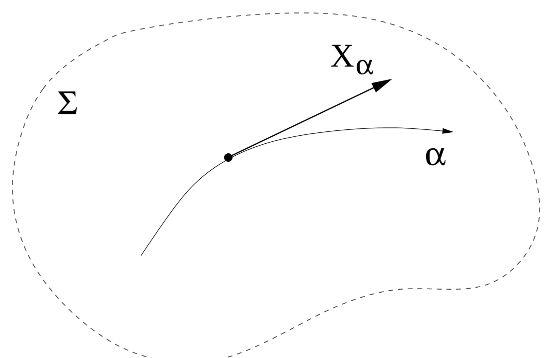
$$\langle\, n \mid (T\bar{T}) \mid n\,\rangle = -\frac{\pi^2}{R}\left(E_n(R)\frac{d}{dR}E_n(R) + \frac{1}{R}\,P_n^2(R)\right).$$

$T\bar{T}$ flow

Consider a curve $\mathcal{A}_\alpha$ in Σ, with α being the parameter along the curve, such that at each point the tangent vector to the curve is $\sim X = (T\bar{T})$,

$$\frac{d\mathcal{A}_\alpha}{d\alpha} = \frac{1}{\pi^2}\int (T\bar{T})_\alpha(z)\, d^2z\,,$$

where the subscript in $X_\alpha \equiv (T\bar{T})_\alpha$ is added to emphasize that the field belongs to the QFT $\mathcal{A}_\alpha$:



I call the curves $\mathcal{A}_\alpha$ the "$(T\bar{T})$ flow".

Remark: Note that $(T\bar{T})$ is "irrelevant" in the RG sense (by construction, it has exact dimension 4). Usually, perturbing with an RG irrelevant operator does not unambiguously define a theory, as one has to add along a tower of largely undetermined "counterterms" of yet higher dimensions. In the case in question the defining equation $d\mathcal{A}_\alpha/d\alpha = \frac{1}{\pi^2}\int (T\bar{T})_\alpha(z)\, d^2z$ can be regarded as an infinite set of "normalization conditions" which fully define all the counterterms.

Since at any point of the curve

$$\frac{\partial E_n(R,\alpha)}{\partial\alpha} = \langle n \mid \int (T\bar{T})_\alpha(z)\, d\mathsf{x} \mid n\rangle = R\,\langle n \mid (T\bar{T})_\alpha \mid n\rangle_\alpha$$

one arrives at closed differential equation

$$\frac{\partial}{\partial\alpha}E(R,\alpha) + E(R,\alpha)\,\frac{\partial}{\partial R}E(R,\alpha) + \frac{P^2(R)}{R} = 0\,.$$

where I've dropped the index n in

$$E(R,\alpha) = E_n(R,\alpha)$$

because the equation is the same for all levels. The equation has the form of equation of motion for compressible inviscid fluid in 1D - the (inviscid) Burgers equation - with the "driving force" $-P^2/R = (2\pi k)^2/R^3$. Given $E(R,0)$ one can determine $E(R,\alpha)$ at all α along the flow.

This equation can be used to derive a number of other results, including exact α dependence of the particle scattering amplitudes. In the limit $R \to \infty$ the finite size spectrum defines the density of states, and thus the S-matrix. For example, the elastic $2 \to 2$ element deforms as

$$S_{2\to 2}(s,\alpha) = S_{2\to 2}(s,\alpha)\, e^{-i\alpha\, \sqrt{s(s-4M^2)}} \;\sim\; S_{2\to 2}(s,\alpha) e^{-i\alpha\, E_{CM}^2}$$

Note the abnormally fast growth of the additional phase at high energy.

Deformations of other S-matrix elements are given by similar but more complicated formulae.

Important general conclusion can be made about the UV behavior of the ground state energy $E_\alpha(R) = E(R, \alpha)$. For the ground state $P = 0$, and the Burgers equation admits elementary solution

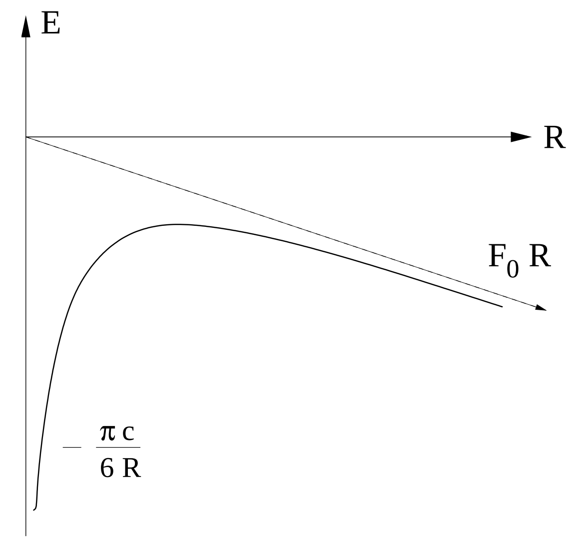
$$E_\alpha(R) = E_0\,(R - \alpha\, E_\alpha(R))\ ,$$

which takes even more transparent form in terms of the functions $R_\alpha(E)$ and $R_0(E)$, inverse to $E_\alpha(R)$ and $E_0(R)$, respectively:

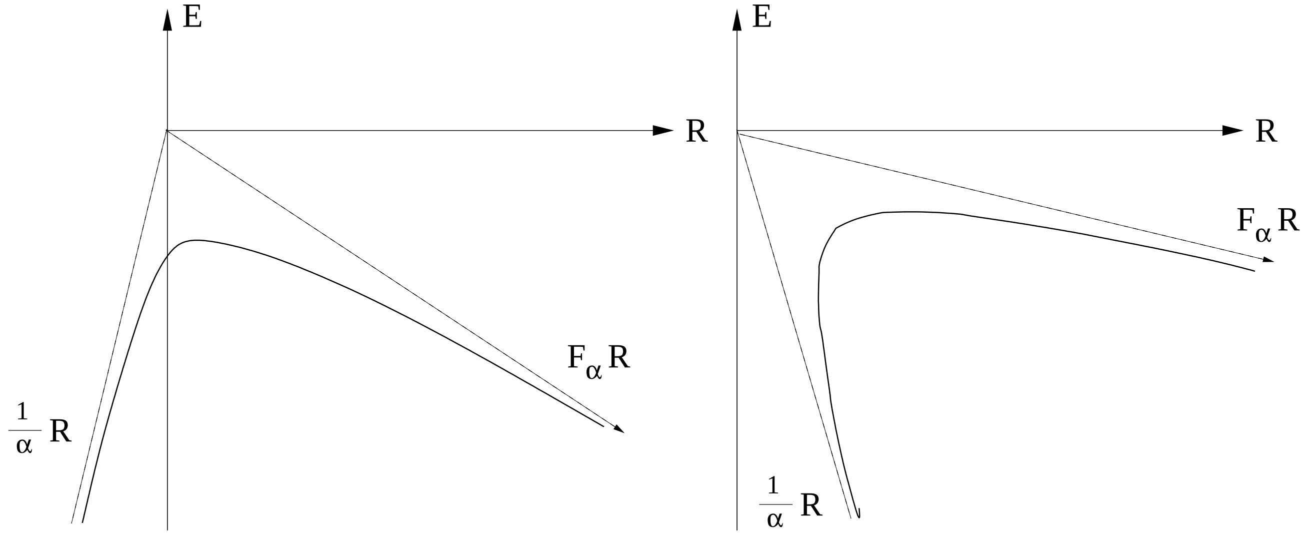
$$R_\alpha(E) = R_0(E) + \alpha\, E\, .$$

The plot of $R_\alpha(E)$ is related to the plot of $R_0(E)$ by the affine transformation $E \to E,\ R \to R + \alpha E$.

Assume that $\mathcal{A}_0$ is UV complete QFT, whose UV limit is controlled by CFT with the central charge $c > 0$. Then the plot of $E_0(R)$ looks, qualitatively, like this



At large R it approaches the linear form $\mathcal{E}\, R$, with $\mathcal{E}$ being the bulk vacuum energy density, while as $R \to 0$ it has the standard CFT behavior. The above affine transformation reveals the following form(s) of $E_\alpha(R)$



where the left plot applies to the case $\alpha > 0$, while the right plot corresponds to $\alpha < 0$.

There are reasons to believe that, when originated at UV complete $\mathcal{A}_0$, the $(T\bar{T})$ flow is ill-defined (has no ground state) at $\alpha > 0$ (I will bring up some argument later in this talk). Therefore, here I concentrate on the case of negative α. As the plot shows, in this case $E_\alpha(R)$ develops square-root singularity at some positive $R = R_*$, so that continuation to $R < R_*$ returns complex values.

For example, if $\mathcal{A}$ is itself a CFT, i.e.

$$E_0(R) = \mathcal{E}_0\, R - \frac{\pi c}{6R}\,,$$

then

$$E_\alpha(R) = \mathcal{E}_\alpha\, R + \frac{R}{2\tilde{\alpha}}\left(1 - \sqrt{1+t}\right)\,,$$

where $\mathcal{E}_\alpha = \mathcal{E}_0/(1+\alpha\mathcal{E}_0)$, $\tilde{\alpha} = \alpha\,(1+\alpha\mathcal{E}_0)$, and

$$t = \frac{2\pi c\,\alpha}{3\,R^2}\,.$$

The singularity occurs at $t=-1$, i.e. at

$$R_* = \sqrt{-\frac{2\pi c\alpha}{3}}\,.$$

Apparently, some sort of instability develops at the scales $< R_*$, and it seems important to understand the mechanism behind this instability. Anyway, it seems that the flow $\mathcal{A}_\alpha$ originating from UV complete $\mathcal{A}_0$ is not UV complete in usual QFT sense.

It is likely that the singularity at finite R_* signals breakdown of "local structure". One might think that the "bad" short distance behavior, including the problem with locality, is related to too fast high energy growth of the scattering phase

$$\delta = -i\alpha \sinh\theta \sim -i\alpha\, E_{CM}^2\,.$$

(normally disapproved in QFT). But arguments exist that the phenomenon might be of different, and more general, nature.

The arguments are based on interesting extension of the notion of $(T\bar{T})$ flow, which exists in the special case of <u>integrable</u> QFT.

Integrable QFT (IQFT)

One of the common properties of all known IQFT is the presence of an infinite set of higher-spin local Integrals of Motion (IM)

$$P_s = \frac{1}{2\pi} \int_C T_{s+1}(z)\, d\mathsf{z} + \Theta_{s-1}(z)\, d\bar{\mathsf{z}}$$
$$\bar{P}_s = \frac{1}{2\pi} \int_C \bar{T}_{s+1}(z)\, d\bar{\mathsf{z}} + \bar{\Theta}_{s-1}(z)\, d\mathsf{z}$$

where (T_{s+1}, Θ_{s-1}) and $(\bar{T}_{s+1}, \bar{\Theta}_{s-1})$ are components of local currents which satisfy the continuity equations

$$\partial_{\bar{\mathsf{z}}} T_{s+1}(z) = \partial_{\mathsf{z}} \Theta_{s-1}(z)\,, \qquad \partial_{\mathsf{z}} \bar{T}_{s+1}(z) = \partial_{\bar{\mathsf{z}}} \bar{\Theta}_{s-1}(z)\,.$$

The index s (associated with the Lorentz spin of the IM) runs certain subset $\{s\} \subset \mathbb{Z}_+$, characteristic of the IQFT. The IM all commute

$$[P_s, P_{s'}] = [P_s, \bar{P}_{s'}] = [\bar{P}_s, \bar{P}_{s'}] = 0\,.$$

From these continuity equations, exactly as in the case of the Energy-Momentum tensor, one can derive the relations

$$T_{s+1}(z)\bar{T}_{s-1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z') = X_s(z') + \text{derivatives}\,,$$

which define, up to derivatives, the scalar fields $X_s(z)$ of exact dimensions $2s$. Thus, it defines X_s as vectors in $T\Sigma|_{QFT}$. Moreover, it can be shown that the infinitesimal deformations

$$\mathcal{A} \;\to\; \mathcal{A} + \sum_{s\in\{s\}} \delta\alpha_s \int X_s(z)\, d^2 z$$

preserves all the IM P_s, $\bar{P}_s$. Starting from some IQFT $\mathcal{A}_0$, one can integrate the infinitesimal deformations into infinite-dimensional subspace $\Sigma^{\text{Int}} \subset \Sigma$, with local coordinates α_s, so that

$$X_s \in T\Sigma^{\text{Int}}|_{\text{IQFT}}\,.$$

Thus, in the integrable case, the notion of $(T\bar{T})$ flow can be extended to an infinite-dimensional "flow" in α_s.

If IQFT is massive, it can be uniquely associated with a factorizable S-matrix. The full S-matrix is expressed in terms of the

$$2 \to 2 \quad \text{amplitude} \quad \widehat{S}(\theta)\,, \qquad \theta = \theta_1 - \theta_2$$

(can be an operator in the particle's "flavor" spaces). The latter satisfies a number of general conditions (unitarity, analyticity/crossing, bootstrap, Yang-Baxter equation). The conditions fix $\widehat{S}(\theta)$ up to the "CDD factor", i.e. leaves the ambiguity

$$\widehat{S}(\theta) \;\to\; \widehat{S}(\theta)\,\Phi(\theta)\,,$$

where the factor Φ is to satisfy

$$\left\{ \begin{array}{c} \Phi(\theta) = \Phi(i\pi - \theta) \\ \Phi(\theta)\Phi(-\theta) \;=\; 1 \end{array} \right\} \qquad \Rightarrow \qquad \Phi(2\pi i + \theta) = \Phi(\theta)\,,$$

plus possibly additional constraints from the bound-state structure (the bootstrap conditions).

Formal but general representation

$$\Phi(\theta) = \exp\Big\{-i \sum_{s\in\{s\}} \alpha_s \sinh(s\theta)\Big\}$$

(converges for sufficiently small θ, and defined by analytic continuation beyond that domain). Here s runs positive integers, but the bootstrap conditions limit the admitted values to some infinite subset $\{s\} \subset \mathbb{Z}_+$, which always coincides with the set of spins of the local IM P_s. The curve $\{\alpha_s = 0, s > 1; \alpha_1 = \alpha\}$ reduces to the $(T\bar{T})$ flow. Generally

Infinitesimal CDD deformations of $\widehat{S}$	$\leftrightarrow$	$\{X_s\}$ deformations of $\mathcal{A}_{\mathrm{IQFT}}$

If only finitely many α_s are involved, the above scattering phase has even worse high energy behavior. However, generic solution for the CDD factor does not need to have "bad" high-energy limit.

Alternative representation of generic CDD factor if (a compactification of) the form

$$\Phi(\theta) = \prod_{p=1}^{N} \frac{B_p - i\,\mathsf{sinh}\,\theta}{B_p + i\,\mathsf{sinh}\,\theta}\,, \qquad \{B\} = \cup_N \{B_p\}\,.$$

Note that for finite N such CDD factors have normal high energy behavior.

It is expected that, just as in Σ itself, majority of points in Σ^{Int} do not define UV complete QFT, with local structure and all. On the other hand, as was observed, all elements of Σ^{Int} are in correspondence with factorizable S-matrices. S-matrix determines, in principle, all physical content of the theory, and no UV cutoff needs to be introduced. Then, how the Landau-Abrikosov-Khalatnikov phenomenon shows up in the S-matrix approach?

Consider again the ground state in a finite size geometry of a cylinder of circumference R. In integrable theories, given $\widehat{S}(\theta)$, one can systematically compute the ground state energy $E_{\mathsf{vac}}(R)$ using the "Thermodynamic Bethe Ansatz" (TBA) equation.

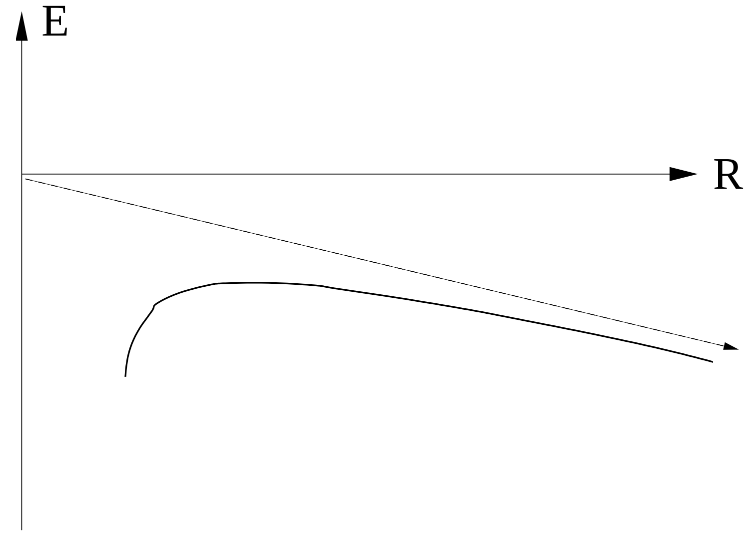
TBA is a device which, with the input of $\widehat{S}(\theta)$, computes (in most cases numerically) the vacuum energy $E_{\text{vac}}(R)$,

$$\text{TBA}: \qquad \widehat{S}(\theta) \quad \to \quad E_{\text{vac}}(R)\,.$$

Then one can taste the behavior of $E_{\text{vac}}(R)$ for arbitrary CDD factor $\Phi(\theta)$. Technically, one can start with the product forms of the CDD factors, with small number N of the CDD factors, and then go up in N.

No systematic analysis was ever done, to the best of my knowledge. However, preliminary calculations of this sort was done in early 90th by Al. Zamolodchikov, inspired by the "staircase model". Several sample CDD factors were tested (with limited number of factors). The results could be summarizes as follows:

Unless the parameters B_p are fine-tuned to special values where UV complete behavior is observed (e.g. the "staircase model"), the function $E_{\text{vac}}(R)$ develops a singularity at some finite R_* ,



Moreover, high precision calculation at R close to R_* reveals that in all cases the singularity is a square root branching point.

• These universal character of the singularity suggests the common mechanism of developing of instability. Note that solution for the $(T\bar{T})$ flow exhibits the same singularity.

• Since the finite N CDD do not have abnormal high-energy behavior, it is unlikely that the UV problem is related to the too fast growth of the scattering phase, and probably has much more general nature.

Semiclassical analysis:

Some insight can be gained in the case when

- $\mathcal{A}_0 = CFT$.

- $c \to \infty$, so that $T, \bar{T}$ are classical fields.

I assume that (i) The ground state is determined by some classical configuration $T_{cl}(z), \bar{T}_{cl}(z)$, and (ii) $T_{cl}(z)$ and $\bar{T}_{cl}(z)$ are constants, independent of z. In the following discussion I drop the subscript cl under $T, \bar{T}$.

Generally, the currents T_{s+1}, $s = 2n-1$ are polynomials in T and its derivatives,

$$T_{s+1} = T^n + a_1 \, T^{n-3}(T')^2 + \ldots ,$$

and when the classical configuration is constant, the derivatives can be dropped, so that

$$T_{2n} = T^n \, , \qquad X_{2n-1} = (T\bar{T})^n \, .$$

Therefore, general X_s-deformed action (for the purpose of the ground state calculation) can be replaced by

$$\mathcal{A} = \mathcal{A}_{CFT} + \sum_{n=1}^{\infty} \alpha_{2n-1} \int (T\bar{T})^n \, d^2 z = \mathcal{A}_{\mathsf{CFT}} + \frac{1}{\alpha} \int U(\alpha^2 \, T\bar{T}) \, d^2 z$$

where I wrote $\alpha_{2n-1} = \alpha^{2n-1} \, C_n$ with dimensionless C_n, the Taylor coefficients of U. When α_s are finite, the currents T_{s+1}, Θ_{s-1}, etc, receive α-dependent corrections, but clearly for constant $T, \bar{T}$ this structure is preserved.

Now, introducing the auxiliary fields $\mu, \bar{\mu}$, one can further replace the above action with

$$\mathcal{A}_\mu - \frac{4}{\alpha} \int W(\mu\bar{\mu}) \, d^2 z \, , \qquad \mathcal{A}_\mu = \mathcal{A}_{\mathsf{CFT}} + \frac{1}{\pi} \int (T\mu + \bar{T}\bar{\mu}) \, d^2 z \, ,$$

where $W(\mu\bar{\mu})$ is the double (in T and $\bar{T}$) Legendre transform of U.

For constant $\mu, \bar{\mu}$

$$\langle \, e^{-\mathcal{A}_\mu} \rangle_{\mathsf{CFT}} = e^{-E_\mu \, L} \, ,$$

$$E_\mu = -\frac{\pi c}{6R}\left[\frac{1}{1+\mu} + \frac{1}{1+\bar{\mu}} - 1\right],$$

where I assumed the geometry of a cylinder of the circumference R and the length L. Finally, the ground state energy is obtained by minimizing the function

$$E(\mu,\bar{\mu}) = -\frac{R}{\alpha}\left[\frac{t}{1+\mu} + \frac{t}{1+\bar{\mu}} - t + W(\mu\bar{\mu})\right]$$

with respect to μ and $\bar{\mu}$. Here again

$$t = \frac{2\pi c\,\alpha}{3R^2}\,.$$

The problem deserves systematic analysis with different forms of W, which is under way. However, it is possible to identify W associated with the $(T\bar{T})$ flow. It turns out

$$W_{T\bar{T}\ \text{flow}}(\mu\bar{\mu}) = \frac{\mu\bar{\mu}}{1-\mu\bar{\mu}}\,.$$

With this form the above minimization problem yields the vacuum energy

$$E_{\text{vac}} = -\frac{R}{2\alpha}\left(\sqrt{1+t} - 1\right),$$

which is exactly our result for the $(T\bar{T})$ flow from the Burgers equation.

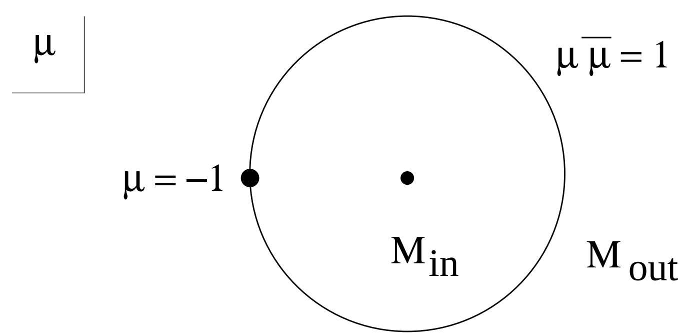
It is interesting to observe the mechanism behind the singularity formation in this model. Looking again at the energy function

$$E(\mu,\bar{\mu}) = -\frac{R}{\alpha}\left[\frac{t}{1+\mu} + \frac{t}{1+\bar{\mu}} - t + \frac{\mu\bar{\mu}}{1-\mu\bar{\mu}}\right]$$

we observe that it has two parts: the "CFT part" which depends on μ and $\bar{\mu}$ separately, and the "deformation term", the function of the product $\mu\bar{\mu}$ only. The singularity at $\mu\bar{\mu} = 1$ divides the configuration space into two distinct domains,

$$M_{\text{in}}: \quad \mu\bar{\mu} < 1 \qquad \text{and} \qquad M_{\text{out}}: \quad \mu\bar{\mu} > 1\,,$$

separated by an infinite "barrier".



It is natural to assume that the relevant domain is M_{in}, since it is the one that contains the point $(\mu, \bar{\mu}) = 0$, the saddle point at $\alpha = 0$. Then, obviously, with positive α the energy function is unbounded from below in the M_{in} at any R, suggesting that at $\alpha > 0$ the theory is not defined as the path integral.

At $\alpha < 0$, instead, the W-term is bounded from below in M_{in}, and it grows indefinitely as one approaches the "boundary". However, the other terms are singular at special point $(\mu, \bar{\mu}) = (-1, -1)$ on the boundary, and diverge to $-\infty$ as one approaches this point from M_{in}. Competition of these two term makes $E(\mu, \bar{\mu})$ bounded from below at $R > R_*$, but it looses the lower bound at $R < R_*$. Thus, at $R < R_*$ the theory is unlikely to have a ground state.

It would be important to analyze fluctuations over this classical solutions, in particular understand mechanism of decay at $R < R_*$.

In general semiclassical models, it is tempting to explore various types of UV behavior which emerges under different choices of W. However, it is important to understand how reliable the leading classical analysis is. Also, it is not yet clear how the classical description in terms of the "potential" W relates to the realization via the CDD factor.

STOP

Happy Birthday IM & Many Happy Returns!