Quantization of Hydrodynamics: Hydrodynamics and Gravity

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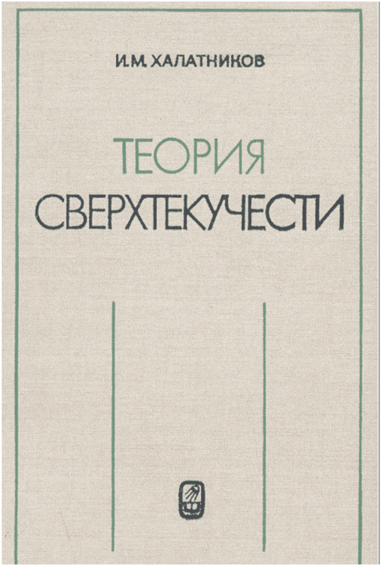
Khalatnikov Centennial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Hydrodynamics is the only remaining Hamiltonian system defying quantization

Intractable problem?

Nature confronts us with with experimentally accessible quantum ideal fluids:

(i) superfluid helium;

(ii) electronic fluid in the FQH regime, Fluid is incompressible $\nabla \cdot u = 0,$

Vorticity $\omega = \nabla \times u,$

Euler equation $\boxed{D_t \mathrm{u} = -\nabla p}$

Material derivative $D_t = \partial_t + u \cdot \nabla$

Helmholtz equation $\boxed{D_t \omega = 0}$ vorticity is frozen into the flow

Conservation Law: $\boxed{\dot{\omega} = -\epsilon_{ik} \partial^k \nabla_j \left(u_i u_j\right)}$

traceless part of momentum flux tensor

Conservation Law:

$$\boxed{\dot{\omega} = -\epsilon_{ik}\partial^k \nabla_j \left(u_i u_j\right)}$$

Quantum corrections

$$\langle \dot{\omega} \rangle = \epsilon_{ik}\partial^k \nabla_j \langle u_i u_j \rangle = 0$$

Quantum stress (Reynolds stress)

$$\langle u_i u_j \rangle = \langle u_i \rangle \langle u_j \rangle - T_{ij}$$

Quantum correction to the Helmholtz equation

$$D_t \langle \omega \rangle = \epsilon_{ik}\partial^k \nabla_j T_{ij}$$

The problem is to compute the quantum stress in terms of vorticity

$$T_{ij}\left(\langle \omega \rangle\right) = -\langle\!\langle u_i u_j \rangle\!\rangle$$

All vortices (or vorticity patches) are sign-like

$$\boxed{\omega > 0}$$

In this case the quantum Reynolds tensor (up to divergence-free part) could be determined exactly as a local function of vorticity $T(\omega)$

In complex notations

$$T = T_{xx} - T_{yy} - 2\mathrm{i}T_{xy}$$

$$T = \frac{\hbar}{24\pi}\left(\partial_z^2 \log\omega - \frac{1}{2}(\partial_z \log\omega)^2\right)$$

Schwarzian of a **metric**

$$ds^2 = \omega|dz|^2, \quad \omega > 0.$$

Curvature of the metric

$$\mathscr{R} = -\omega^{-1}\Delta\log\omega$$

Quantum Helmholtz equation

$$\boxed{D_t\omega = \tfrac{\hbar}{96\pi}\nabla\mathscr{R}\times\nabla\omega.}$$

Quantum Euler equation

$$\dot{u} + (u\cdot\nabla)u + \nabla p = \frac{\hbar}{96\pi}\mathscr{R}\times\Delta u + \dots$$

Correspondence: Chiral flows $\longleftrightarrow$ Riemannian geometry 

Positive vorticity 2-form $\quad \omega_{ij} = \partial_i u_j - \partial_j u_i = \epsilon_{ij}\omega, \quad \omega > 0$

Kähler structure $\quad \omega \mathrm{d}z\mathrm{d}\bar{z}$

Measure $\quad \int D[\text{vorticity}] \longleftrightarrow \int D\,[\text{Riemann metric}]$

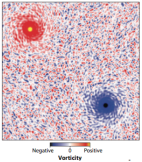
Quantum chiral flows $\longleftrightarrow$ Quantum gravity

Chiral flows: all vortices are sign-like

$$\boxed{\omega > 0}$$

Examples of chiral flows (classical):

- tornado, hurricane, storms;
- Red Spot of Jupiter;
- Onsager clusterization



- Rotating superfluid helium;
- Fractional quantum Hall effect.

Correspondence between rotating

$$n_A = \frac{e}{h} B, \quad \text{number of atoms=number of magnetic flux quanta;}$$

$$n_v = \frac{2\Omega}{\Gamma}, \quad \text{number of vortices=number of electrons;}$$

$$\nu = \frac{n_v}{n_A} \quad \text{filling fraction} = \begin{cases} 10^{-3} & \text{superfluid helium,} \\ 1/3 & \textit{FQHE}. \end{cases}$$

$$\Gamma = \frac{h}{m_A}, \quad \Omega = \text{frequency of rotation.}$$

$$\boxed{D_t\omega = \frac{\hbar}{96\pi}\nabla\mathscr{R}\times\nabla\omega}$$

$$\mathscr{R} = -\omega^{-1}\Delta\log\omega$$

Fundamental Symmetry of fluids:

Relabeling symmetry (invariance with respect to diffeomorphisms);

Relabeling symmetry **uniquely** determines the quantum correction.

Regularization and Relabeling Symmetry

$$D_t\omega \equiv (\partial_t + u\cdot\nabla)\omega = 0;$$
$$\dot{\omega} = -\epsilon_{ik}\partial^k\nabla_j\left(u_i u_j\right).$$

Problem: regularization of the advection term:

Point splitting

$$\lim_{\epsilon\to 0}\left(\mathbf{u}|_{\mathbf{r}-\epsilon}\ \mathbf{u}|_{\mathbf{r}+\epsilon}\right)$$

generally does not respect the relabeling symmetry

UV cut-off depends on the flow

$$\epsilon[\mathbf{u}]$$

Fluid dynamics is Hamiltonian $H = \frac{1}{2} \int u^2 \, \mathrm{d}V,$

Poisson brackets (Landau 1941) $\{\omega(r),\ \omega(r')\} = (\nabla_r \times \nabla_{r'})\, \omega(r) \delta_{rr'}$

Lie-Poisson algebra **sdiff**:

$$\{\omega(r),\ \omega(r')\} = (\nabla_r \times \nabla_{r'})\omega(r)\delta_{rr'}$$

$$\omega(r) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\omega_{\mathbf{k}}$$

$$\boxed{\{\omega_{\mathbf{k}},\ \omega_{\mathbf{k}'}\} = (\mathbf{k} \times \mathbf{k}')\omega_{\mathbf{k}+\mathbf{k}'}}$$

- Quantization: $\{\omega,\ \omega\} \to \frac{1}{i\hbar}\,[\omega,\ \omega],$

- Lie algebra: **area preserving diffeomorphisms SDiff**

$$\boxed{[\omega_{\mathbf{k}},\ \omega_{\mathbf{k}'}] = i\hbar\,(\mathbf{k}\times\mathbf{k}')\,\omega_{\mathbf{k}+\mathbf{k}'}}$$

- Fluid dynamics is the action of the group of **SDiff**

- Flows are points of coadjoint orbits of **SDiff**

Problem:
- Representations of **SDiff** are not known (despite many attempts)

Vortices form the finite dimensional representation of **SDiff**

Circulation of vortices in quantum fluids is quantized

$$\omega = \sum_{i=1}^{N} \Gamma_i \delta(z - z_i)$$

Poisson brackets

$$\{\bar{z}_i,\ z_j\} = 2(\Gamma_i)^{-1}\delta_{ij}$$

KIrchhoff Hamiltonian

$$H = -\sum_{i \neq j} \Gamma_i \Gamma_j \log|z_i - z_j|$$

$$\boxed{\dot{\bar{z}}_i = \frac{\mathrm{i}}{2\pi} \sum_{j \neq i} \frac{\Gamma_j}{z_i - z_j}.}$$

The choice of Lagrangian coordinates:

pathlines of fluid particles : $\int D\vec{X}_t \; e^{-\frac{i}{\hbar}\text{Action}}$

pathlines of vortices : $\int Dz_t D\bar{z}_t \; e^{-\frac{i}{\hbar}\text{Action}}$

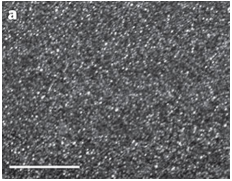
Large number of same sense vortices

$$\Gamma_i = \Gamma = 2\pi\hbar$$

Vorticity is the number of vortices:

$$\text{Fraction}: \quad \nu = \frac{\text{number of vortices}}{\text{number of atoms}}$$

$$\bar{z}_i = z_i^\dagger, \quad [\bar{z}_i,\, z_j] = 2\nu\ell^2\delta_{ij}$$



Observed vortices in He^4

Quantum vorticity operator

$$\omega_k = \int e^{\mathbf{i}\mathbf{k}\cdot\mathbf{r}}\omega(r)d^2r = \sum_i e^{-\frac{\mathrm{i}}{2}\mathrm{k}\bar{z}_i^\dagger} e^{-\frac{\mathrm{i}}{2}\bar{\mathrm{k}}z_i}$$

Quantum vorticity operator

$$\omega_k = \sum_i e^{-\frac{i}{2}\mathrm{k}\bar{z}_i^{\dagger}} e^{-\frac{i}{2}\bar{\mathrm{k}}z_i}, \quad [\bar{z}_i,\ z_j] = 2\nu\ell^2\delta_{ij}$$

Sine algebra: deformation of $\mathbf{SDiff}(\mathbf{T}^2)$

$$[\omega_k,\ \omega_{k'}] = e_{kk'}\omega_{k+k'}$$

Structure constants

$$e_{kk'} = 2\mathrm{i}e^{\left(\frac{\mathbf{k}\cdot\mathbf{k}'}{4\pi N_{\mathrm{A}}}\right)} \sin\left(\frac{\mathbf{k}\times\mathbf{k}'}{4\pi N_{\mathrm{A}}}\right)$$

What is the algebra of the stress

$$T = -\sum_k e^{i(k+k')\cdot r}(\mathbf{k}\mathbf{k}')^{-1}\omega_{\mathbf{k}}\omega_{\mathbf{k}'}$$

$$\{\omega_k,\ \omega_{k'}\} \xrightarrow{\hbar} [\omega_k,\ \omega_{k'}] = e_{kk'}\omega_{k+k'}$$

$$e_{k,k'} = i\mathbf{k}\times\mathbf{k}' \xrightarrow{N} 2ie^{\left(\frac{\mathbf{k}\cdot\mathbf{k}'}{4\pi N_A}\right)}\sin\left(\frac{\mathbf{k}\times\mathbf{k}'}{4\pi N_A}\right)$$

Two deformation parameters: $\hbar \to 0, \quad N \to \infty$

$$\checkmark \quad N \to \infty,\ \hbar \to 0; \quad N\hbar = \text{fixed}$$

Virasoro-Bott cocycle

$$T(z) = -\sum_n (z-z')^{-n-2} L_n$$

$$[L_n,\ L_m] = (n-m)L_{n+m} + \frac{c=1}{12}(n^3-n)\delta_{n+m,0}.$$

Heizenberg equations

$$\dot{\bar{z}}_i = \frac{\mathrm{i}}{2\pi} \sum_{j \neq i} \frac{\Gamma}{z_i - z_j}, \quad [\bar{z}_i,\ z_j] = 2\nu\ell^2\delta_{ij}$$

Schroedinger equation (stationary fow)

$$\dot{\bar{z}}_i|0\rangle = -\mathrm{i}\Omega\bar{z}_i|0\rangle = \nu(\Gamma/2\pi\mathrm{i})\partial_{z_i}|0\rangle$$

$$\left(\nu\partial_{z_i} - \sum_{j \neq i} \frac{1}{z_i - z_j}\right)|0\rangle = 0.$$

$$|0\rangle = \prod_{i>j}(z_i - z_j)^{1/\nu}, \quad \nu = \frac{\#\text{vortices}}{\#\text{atoms}}$$

Euler equation

$$\dot{u} + (u \cdot \nabla)\, u = -\nabla p, \quad \nabla \cdot u = 0$$

Kirchhoff Equation

$$\mathrm{u}(z) = \sum_i \frac{\Gamma_i}{z - z_i}, \qquad \dot{\bar{z}}_i = \frac{\mathrm{i}}{2\pi} \sum_{j \neq i} \frac{\Gamma_j}{z_i - z_j}.$$

Stochastic Kirchhoff equation: noise acting on vortices (not on fluid)

$$d\bar{z}_i = \frac{i}{2\pi}\sum_{j\neq i}\frac{\Gamma_j dt}{z_i - z_j} + d\bar{B}_i$$

$$\text{viscosity} \quad \mathbb{E}\left[d\bar{B}_i dB_j\right] = \nu\delta_{ij}dt$$

This leads to Navier-Stokes equation

$$\dot{u} + (u\cdot\nabla)u + \nabla p = \nu\Delta u + \dots$$

"Symplectic noise": non-dissipative noise

$$\text{odd-viscosity} \quad \mathbb{E}\left[dB_i dB_j\right] = 2i\hbar\delta_{ij}dt$$

Non-dissipative stochastic flows equivalent to quantization:

$$\dot{u} + (u\cdot\nabla)u + \nabla p = \frac{\hbar}{96\pi}\mathcal{R}\times\Delta u + \dots$$