Motion of complex singularities and integrability of fully nonlinear free surface dynamics of superfluid Helium vs. single ideal fluid

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Support:
Russian Scientific Foundation 14-22-00259, NSF 0807131, NSF 1004118, NSF 141214, NSF 1814619

Happy Birthday and Anniversary to Isaak Markovich!

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3D Euler's equations of incompressible fluid motion in gravitational field *g*

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p + \mathbf{g} = 0$$

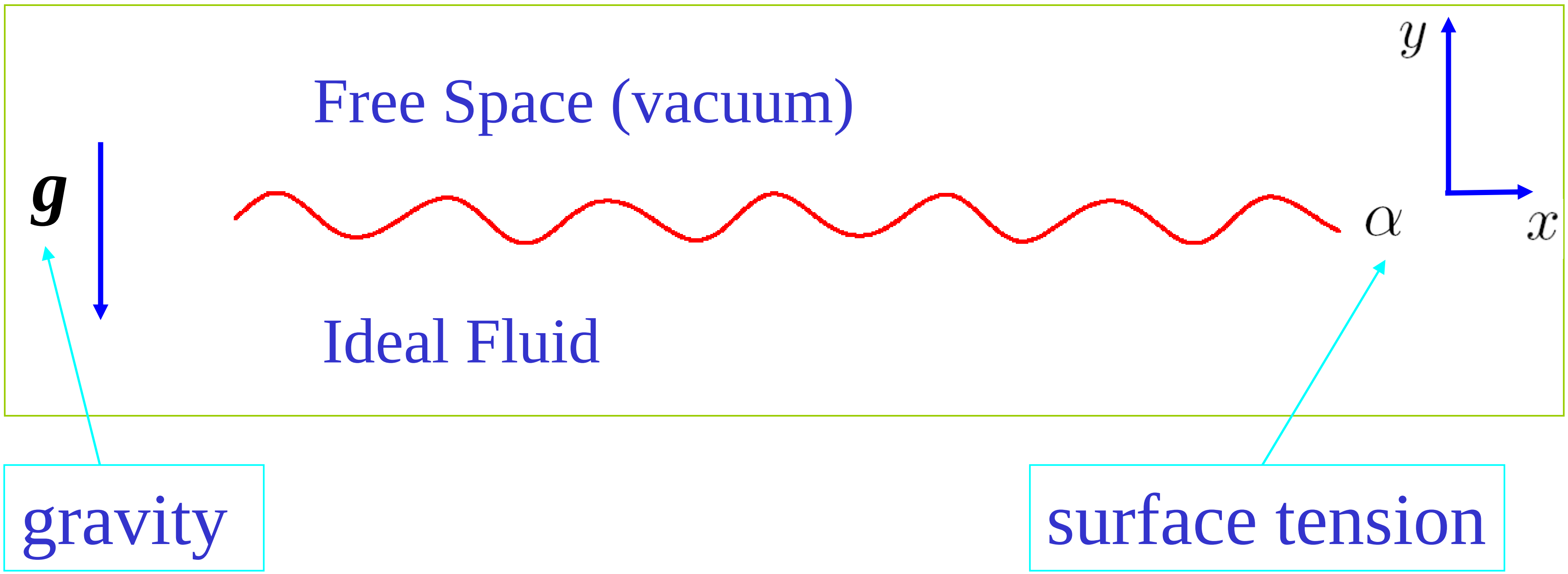
$$\nabla\cdot\mathbf{v} = 0$$

Reduction: potential flow

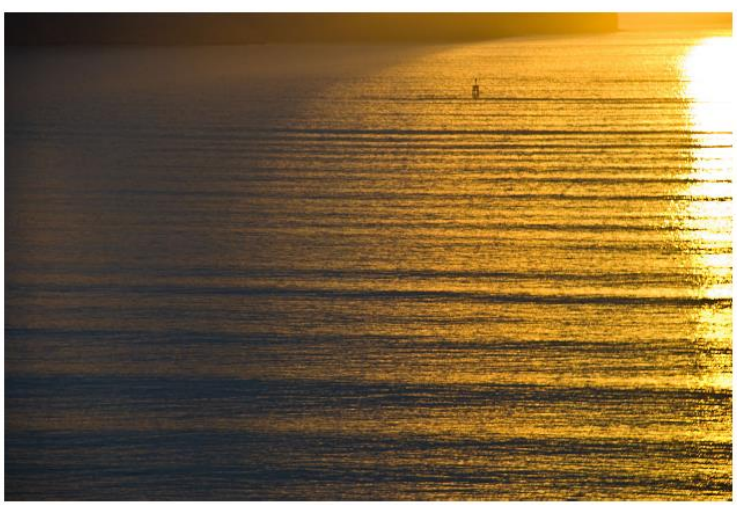
$$\mathbf{v} = \nabla\Phi \qquad \Longrightarrow \qquad \nabla\cdot\mathbf{v} = \Delta\Phi = 0$$ - Laplace equation

$$\nabla\left[\Phi_t + \frac{(\nabla\Phi)^2}{2} + \frac{p}{\rho} + \mathbf{g}\cdot\mathbf{r}\right] = 0$$ - Bernouilli equation

2D Hydrodynamics of ideal fluid with free surface

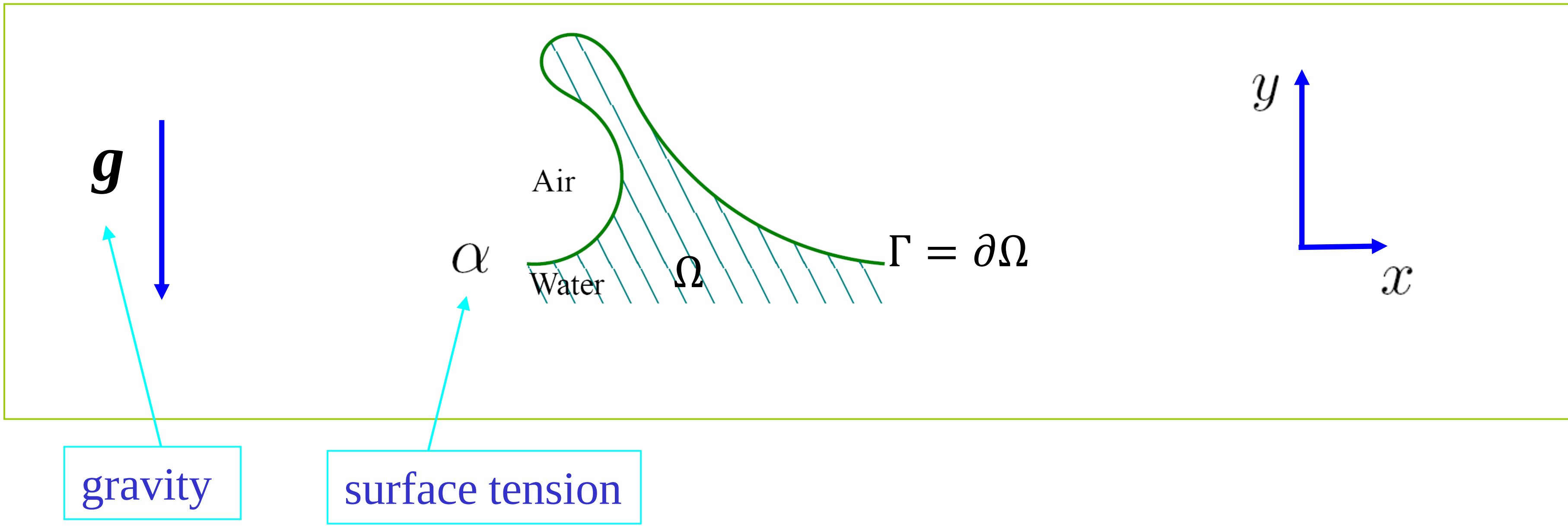


$y = \eta(x,t)$ - natural parametrization of the shape of the free surface for non-overturning waves



- nearly 2D flow (swell) in ocean

2D Hydrodynamics of ideal fluid with free surface for arbitrary strong waves



$y = \eta(x,t)$ - shape of free surface if no overturning (single-valued function)

$(x(u,t), y(u,t))$ - general parametric form of free surface $\Gamma = \partial\Omega$ (instead of multiple-valued $y = \eta(x,t)$):

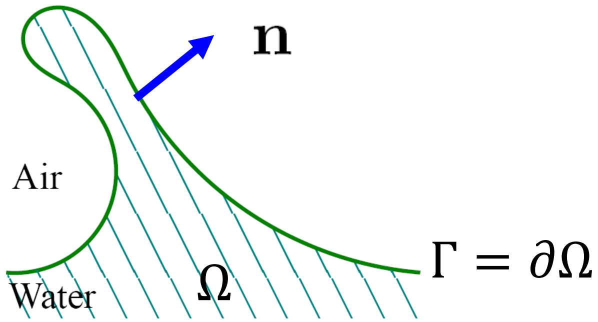
$-\infty < u < \infty$

Boundary conditions at free surface:

Kinematic boundary condition: fluid's free surface moves with fluid particles

$$\mathbf{n} \cdot (x_t, y_t) = V_n \equiv \mathbf{n} \cdot \nabla \Phi|_{x=x(u,t),\ y=y(u,t)}$$

$$\mathbf{n} = \frac{(-y_u, x_u)}{(x_u^2 + y_u^2)^{1/2}}$$ - unit normal vector to the surface



Dynamic boundary condition: pressure jump at the free surface compare with zero pressure outside of fluid (neglecting air density)

$$p|_\Gamma = -\frac{\alpha(x_u y_{uu} - x_{uu} y_u)}{|z_u|^3} \left(= -\alpha \frac{\partial}{\partial x}[\eta_x (1 + \eta_x^2)^{-1/2}] \text{ for } y = \eta(x,t) \right)$$

pressure just under the free surface $\Gamma = \partial\Omega$

Bernouilli equation at the free surface

$$\Phi_t + \frac{1}{2}(\nabla \Phi)^2 + p + gy = 0$$

Kinematic and dynamic boundary conditions together with Laplace equation $\Delta\Phi = 0$ form a closed set of equations.

Equivalent Hamiltonian formulation (V. Zakharov, 1968[1]) for single-valued surface parametrization:

$$\frac{\partial\Psi}{\partial t} = -\frac{\delta H}{\delta\eta},$$
$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi},$$

where $\Phi\big|_{y=\eta} \equiv \Psi(x,t)$ - velocity potential at free surface (i.e. Dirichlet boundary condition)

[1] V.E. Zakharov, J. Appl. Mech. Tech. Phys. **9** (2), 190 (1968).

The Hamiltonian =kinetic energy+ potential energy, $H = K + U$

$$K = \frac{1}{2}\int dx \int_{-\infty}^{\eta} (\nabla\Phi)^2 dy$$

$$U = \frac{1}{2}g\int \eta^2\, dx + \alpha\int \left[\sqrt{1+(\nabla\eta)^2} - 1\right] dx$$

potential energy in the gravitational field

surface tension energy

For general parametrization $(x(u,t), y(u,t))$, converting the Hamiltonian to the integral over free surface and using Green's theorem ($\oint_\Gamma (L\mathrm{d}x + M\mathrm{d}y) = \iint_\Omega (M_x - L_y)\mathrm{d}x\mathrm{d}y$):

(a)

$$g\int_{\Omega} y\,\mathrm{d}x\mathrm{d}y - g\int_{y\le 0} y\,\mathrm{d}x\mathrm{d}y = g\int_\Omega \nabla\cdot\mathbf{F}\,\mathrm{d}x\mathrm{d}y - g\int_{y\le 0} y\,\mathrm{d}x\mathrm{d}y = \frac{g}{2}\int_{-\infty}^{\infty} y^2\,x_u\mathrm{d}u,$$

$$\mathbf{F} = \hat{y}y^2/2$$

(b)

$$K = \frac{1}{2}\int_\Omega (\nabla\Phi)^2\mathrm{d}x\mathrm{d}y = \frac{1}{2}\int_\Gamma V_n\psi\mathrm{d}s = \frac{1}{2}\int_{-\infty}^{\infty} V_n\psi\sqrt{x_u^2+y_u^2}\mathrm{d}u, \quad \mathrm{d}s = \sqrt{x_u^2+y_u^2}\,\mathrm{d}u$$

$$\Rightarrow$$

$$H = \frac{1}{2}\int_{-\infty}^{\infty} V_n\psi\sqrt{x_u^2+y_u^2}\mathrm{d}u + \frac{g}{2}\int_{-\infty}^{\infty} y^2\,x_u\mathrm{d}u + \alpha\int_{-\infty}^{\infty}\left(\sqrt{x_u^2+y_u^2} - x_u\right)\mathrm{d}u$$

Normal velocity component: $V_n = \mathbf{n}\cdot\nabla\Phi$

The Hamiltonian perturbation theory for single-valued parametrization $y = \eta(x,t)$

The Hamiltonian H depend on the normal velocity V_n which has to be expressed in terms of canonical variables Ψ and η.
But $\Phi|_{y=\eta} \equiv \Psi(x,t)$ is the Dirichlet boundary condition for Φ

while V_n is the Neumann boundary condition, $V_n := \mathbf{n} \cdot \nabla\Phi|_{z=\eta}$, for Φ.
It means that we have to solve the Laplace Eq. $\triangle\Phi = 0$ With the Dirichlet $\Phi|_{y=\eta} \equiv \Psi(x,t)$ boundary condition to find V_n .

In other words, it is necessary to determine Dirichlet-Neumann operator[1,2]

$$\hat{G}\psi = V_n = \mathbf{n} \cdot \nabla\Phi|_\Gamma$$

which relates V_n and Ψ .

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[1]V.E. Zakharov, J. Appl. Mech.Tech. Phys. **9** (2), 190 (1968).
[2]W. Craig and C. Sulem. J. Comp. Phys., **108**, 73–83 (1993).

Perturbation technique:

Flat free surface is stable.

Series expansion of V_n in powers of Ψ and η allows to develop a perturbation theory for small deviations from flat surface.

Small parameter of perturbation theory[1]: $|\nabla\eta|$ - a typical slope of surface elevation.

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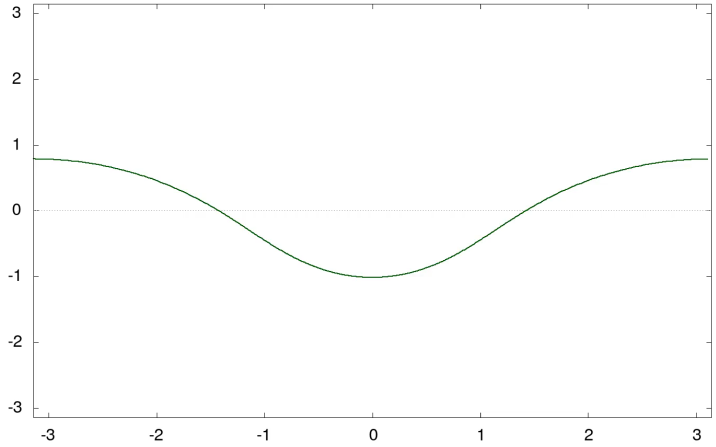
[1] V.E. Zakharov, J. Appl. Mech. Tech. Phys. **9** (2), 190 (1968).

Blow-up and foam formation: strongly nonlinear solutions

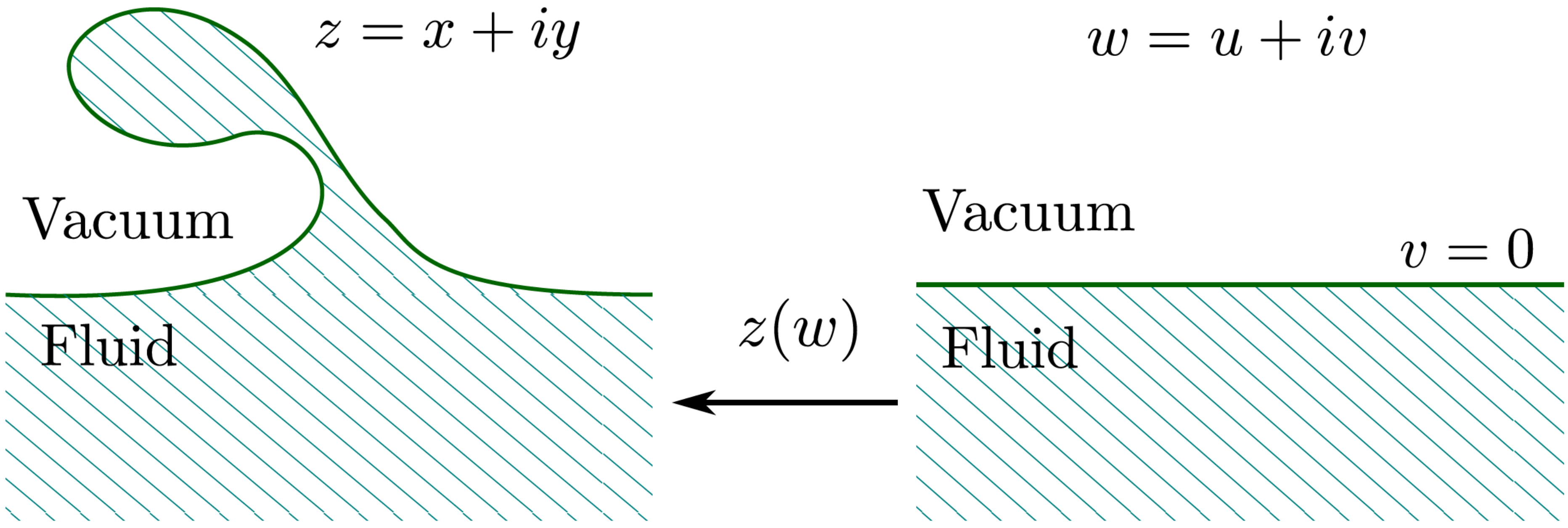


For strongly nonlinear solutions one cannot use the perturbation theory. Instead we use the complex form of 2D hydrodynamics with free surface to explicitly solve the Laplace Eq. $\triangle\Phi = 0$ at each moment of time and find the explicit form of Dirichlet-Neumann operator.

Free surface parametrization in 2D: $(x(u,t), y(u,t))$

Complex variable: $z = x + iy$

Conformal map from lower complex half-plane of $w = u + iv$



Stream function Θ is defined by

$$\frac{\partial}{\partial x}\Theta = -\frac{\partial}{\partial y}\Phi = -v_y \text{ and } \frac{\partial}{\partial y}\Theta = \frac{\partial}{\partial x}\Phi = v_x$$

which ensures the incompressibility condition:

$$\nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}v_x + \frac{\partial}{\partial y}v_y = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\Theta + \frac{\partial}{\partial y}\left[-\frac{\partial}{\partial x}\Theta\right] = 0$$

Define complex potential as $\Pi = \Phi + \mathrm{i}\Theta$

then $\frac{\partial}{\partial x}\Theta = -\frac{\partial}{\partial y}\Phi = -v_y \text{ and } \frac{\partial}{\partial y}\Theta = \frac{\partial}{\partial x}\Phi = v_x$

turns into Cauchy-Riemann equations for analyticity of

$$\Pi(z),\ z = x + \mathrm{i}y$$

The complex velocity: $V := \Pi'(z),\ V = v_x - \mathrm{i}v_y$

Relations between real and imaginary parts of analytical functions

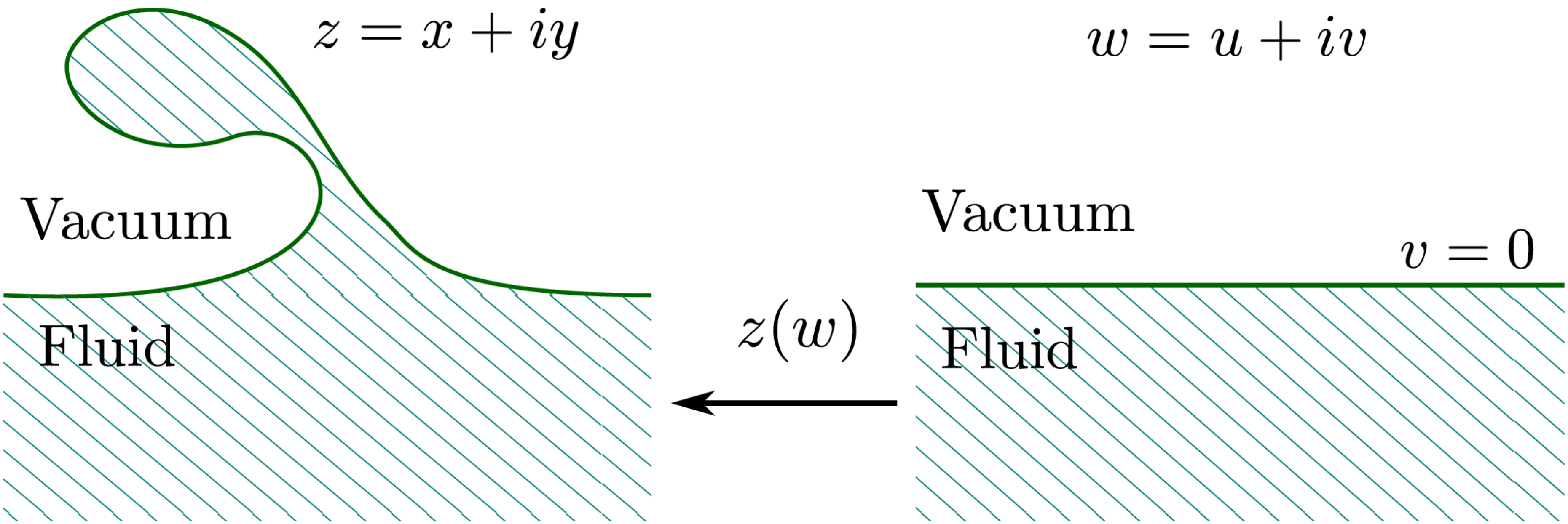
At real line *w*=*u* through Hilbert transform:

$$x = u - \hat{H}y$$
$$\mathit{\Theta} = \hat{H}\mathit{\Phi}$$

$$\hat{H}f(u) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{+\infty}\frac{f(u')}{u'-u}\mathrm{d}u'$$

$$\Pi(z,t) \quad \Rightarrow \quad \Pi(w,t) := \Pi(z(w),t)$$

Analytic in z — Analytic in w



Kinematic boundary condition in conformal variables:

$$\mathbf{n}\cdot(x_t, y_t) = V_n \equiv \mathbf{n}\cdot\nabla\Phi|_{x=x(u,t),\ y=y(u,t)} \qquad \mathbf{n} = \frac{(-y_u, x_u)}{(x_u^2+y_u^2)^{1/2}}$$

$$\Rightarrow \quad \boxed{y_t x_u - x_t y_u = -\hat{H}\psi_u}$$

Dynamic boundary condition in conformal variables:

$$\boxed{\psi_t y_u - \psi_u y_t + g y y_u = -\hat{H}\left(\psi_t x_u - \psi_u x_t + g y x_u\right) - \alpha\frac{\partial}{\partial u}\frac{x_u}{|z_u|} + \alpha\hat{H}\frac{\partial}{\partial u}\frac{y_u}{|z_u|}}$$

Easiest way to obtain: variational principle for the Hamiltonian

$$H = -\frac{1}{2}\int_{-\infty}^{\infty}\psi\hat{H}\psi_u \mathrm{d}u + \frac{g}{2}\int_{-\infty}^{\infty} y^2\left(1-\hat{H}y_u\right)\mathrm{d}u + \alpha\int_{-\infty}^{\infty}\left(\sqrt{(1-\hat{H}y_u)^2+y_u^2} - 1 + \hat{H}y_u\right)\mathrm{d}u$$

Lagrangian $L = \int_{-\infty}^{\infty}\psi(y_t x_u - x_t y_u)\mathrm{d}u - H$ and variation $\delta S = 0$ of

action $S = \int L\mathrm{d}t \qquad \Rightarrow$

Symplectic Hamiltonian form[1]

$$\hat{\Omega}\mathbf{Q}_t = \frac{\delta H}{\delta \mathbf{Q}}, \quad \mathbf{Q} \equiv (y, \psi),$$

Or in components

$$\hat{\Omega} = \begin{pmatrix} \hat{\Omega}_{11} & \hat{\Omega}_{12} \\ \hat{\Omega}_{21} & 0 \end{pmatrix}$$

$$\hat{\Omega}_{11} y_t + \hat{\Omega}_{12} \psi_t = \frac{\delta H}{\delta y},$$
$$-\hat{\Omega}_{12}^{\dagger} y_t = \frac{\delta H}{\delta \psi}.$$

$$\hat{\Omega}_{21} q = x_u q + y_u \hat{H} q = (1 - \hat{H} y_u) q + y_u \hat{H} q$$

$$\hat{\Omega}_{11} q = -\hat{H}(\psi_u q) - \psi_u \hat{H} q, \quad \Omega_{12} q = -x_u q + \hat{H}(y_u q) = -(1 - \hat{H} y_u) q + \hat{H}(y_u q)$$

$$\hat{\Omega}_{11}^{\dagger} = -\hat{\Omega}_{11}$$

$$\hat{\Omega}_{21}^{\dagger} = -\hat{\Omega}_{12} \qquad \langle f, \hat{\Omega}_{ij} g \rangle \equiv \langle \hat{\Omega}_{ij}^{\dagger} f, g \rangle, \, i, j = 1, 2$$

But: not solved for time-derivatives!

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[1]A.I. Dyachenko, P. M. Lushnikov and V. E. Zakharov, Non-Canonical Hamiltonian Structure and Poisson Bracket for 2D Hydrodynamics with Free Surface, J. of Fluid Mech. **869**, 526-552 (2019).

Fluid dynamics in conformal variables (exact form of Euler equation for fluid with free surface)[1]:

$$y_t = (y_u \hat{H} - x_u)\frac{\hat{H}\Psi_u}{|z_u|^2}$$

$$z = x + iy$$

$$\Psi_t = \frac{\hat{H}(\Psi_u \hat{H}\Psi_u)}{|z_u|^2} + \Psi_u \hat{H}\left(\frac{\hat{H}\Psi_u}{|z_u|^2}\right) - gy + \alpha \frac{1}{x_u}\frac{\partial}{\partial u}\frac{y_u}{|z_u|}$$

$$x = u - \hat{H} y$$

Hilbert transform:

$$\hat{H} f(u) = \frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{+\infty} \frac{f(u')}{u' - u}\mathrm{d}u'$$

Hilbert transform in Fourier domain:

$$i\,\mathrm{sign}(k)$$

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[1]A.I. Dyachenko, E.A. Kuznetsov, M. Spector and V.E. Zakharov, Phys. Lett. A **221**, 73 (1996).

Non-canonical Hamiltonian equations for fluid dynamics in conformal variables[1]:

$$\mathbf{Q}_t = \hat{R}\frac{\delta H}{\delta \mathbf{Q}}, \quad \mathbf{Q} \equiv \begin{pmatrix} y \\ \psi \end{pmatrix}, \qquad \hat{R} = \begin{pmatrix} 0 & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{pmatrix}$$

$$y_t = \hat{R}_{12}\frac{\delta H}{\delta \psi},$$

$$\psi_t = \hat{R}_{21}\frac{\delta H}{\delta y} + \hat{R}_{22}\frac{\delta H}{\delta \psi}$$

$$\hat{R}_{11}q = -\psi_u \hat{H}\left(\frac{q}{|z_u|^2}\right) - \frac{1}{|z_u|^2}\hat{H}\left(\psi_u q\right), \quad \hat{R}_{11}^{\dagger} = -\hat{R}_{11},$$

$$\hat{R}_{22}q = 0,$$

$$\hat{R}_{12}q = -\frac{x_u}{|z_u|^2}q - \frac{1}{|z_u|^2}\hat{H}\left(y_u q\right),$$

$$\hat{R}_{21}q = \frac{x_u}{|z_u|^2}q - y_u\hat{H}\left(\frac{q}{|z_u|^2}\right), \quad \hat{R}_{21}^{\dagger} = -\hat{R}_{12}.$$

Poisson bracket

$$\{F,G\} = \sum_{i,j=1}^{2}\int_{-\infty}^{\infty} \mathrm{d}u \left(\frac{\delta F}{\delta Q_i}\hat{R}_{ij}\frac{\delta G}{\delta Q_j}\right) = \int_{-\infty}^{\infty} \mathrm{d}u \left(\frac{\delta F}{\delta y}\hat{R}_{12}\frac{\delta G}{\delta \psi} + \frac{\delta F}{\delta \psi}\hat{R}_{21}\frac{\delta G}{\delta y} + \frac{\delta F}{\delta \psi}\hat{R}_{22}\frac{\delta G}{\delta \psi}\right)$$

Any functionals of *y* commute with each other!

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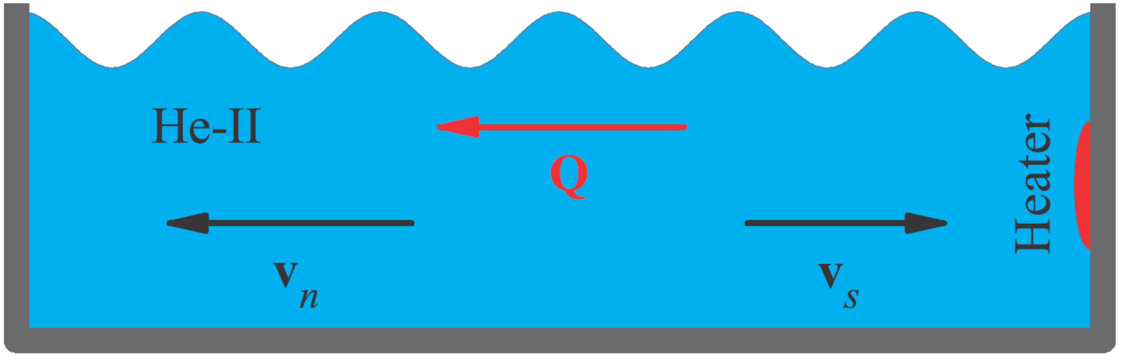
[1] A.I. Dyachenko, P. M. Lushnikov and V. E. Zakharov, Non-Canonical Hamiltonian Structure and Poisson Bracket for 2D Hydrodynamics with Free Surface, J. of Fluid Mech. **869**, 526-552 (2019).

Another form of the Hamiltonian for the same co-symplectic operator:

$$\mathbf{Q}_t = \hat{R}\frac{\delta H}{\delta \mathbf{Q}}, \quad \mathbf{Q} \equiv \begin{pmatrix} y \\ \psi \end{pmatrix}$$

Free surface of superfluid Helium 4[1]: normal component and superfluid component of He 4 shares the same volume of fluid and common free surface



ρ_n and ρ_s - densities of normal and superfluid components

$\mathbf{v}_n$ and $\mathbf{v}_s$ - velocities of normal and superfluid components

Arbitrary nonlinear solutions are exactly integrable in two limits of the viscosity ν_n of normal component:

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[1]P.M. Lushnikov and N.M. Zubarev, Phys. Rev. Lett., **120**, 204504 (2018).

1. Limit of For large kinematic viscosity ν_n : Reduction to two decoupled complex Burgers Eqs in 2D flow[1] (при нулевой вязкости[2])

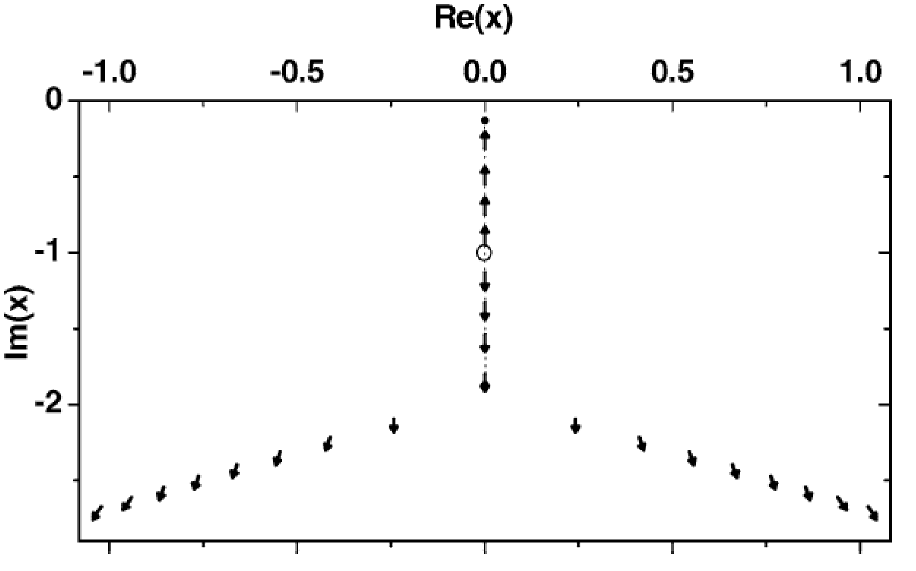
$$\frac{\partial v^{(\pm)}}{\partial t} + v^{(\pm)}\partial_x v^{(\pm)} = \tilde{\nu}\partial_x^2 v^{(\pm)}, \quad \tilde{\nu} = 2\nu_n\rho_n$$

$$v^{(+)} = -2\tilde{\nu}\sum_{j=1}^{n}\frac{1}{x - x_j(t)}.$$

Motion of multiple complex poles:

$$\frac{dx_j}{dt} = -2\tilde{\nu}\sum_{l=1,\, j\neq l}^{n}\frac{1}{x_j - x_l}$$

Particular solution $x_j(t) = i(2\sqrt{\tilde{\nu}t}z_j - a)$,
Where $z_1, z_2, \ldots$ are complex zeros of the Hermite function $H_{\frac{1}{\tilde{\nu}}}(z)$



Interface turns singular if poles reach real axis $x = Re(x)$

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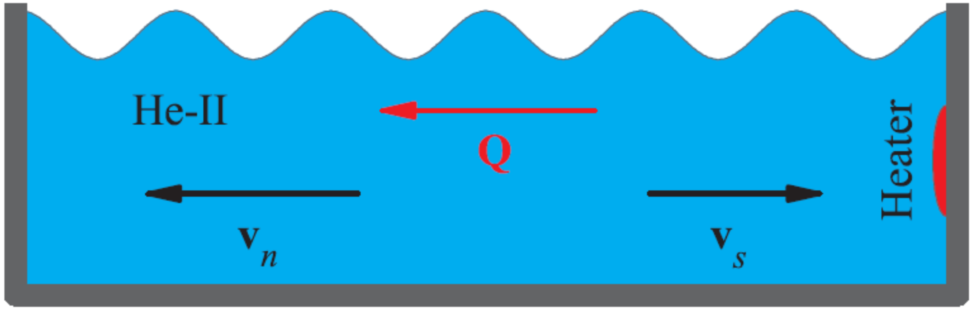
[1]P.M. Lushnikov Phys. Lett. A, **329**, p.49 (2004).

[2]E.A. Kuznetsov, M.D. Spector, and V.E. Zakharov. Phys. Rev. E, **49**:1283–1290 (1994).

2. Non-dissipative two-fluid description of ^{4}He [1,2]

$y = \eta(x,t)$



$$\mathbf{v}_s = \nabla\Phi_s \text{ and } \mathbf{v}_n = \nabla\Phi_n$$

ρ_n and ρ_s - densities of normal and superfluid components

$\mathbf{v}_n$ and $\mathbf{v}_s$ - velocities of normal and superfluid components

$\rho \equiv \rho_s + \rho_n$ - total density

Incompressibility $\Longrightarrow \quad \nabla^2\Phi_{n,s} = 0$

Deep inside Helium: $\Phi_{n,s} \to V_{n,s}\, x$

Assume reference frame with the center of mass: $\rho_n V_n + \rho_s V_s = 0$

Relative velocity between components: $V = V_s - V_n > 0$

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[1]S.E. Korshunov, Europhys. Letters **16**, 673 (1991).
[2]S.E. Korshunov, JETP Letters **75**, 496 (2002).

Boundary conditions at the free surface:

General free surface parametrization in 2D: $(x(u,t), y(u,t))$

$$-\infty < u < \infty$$

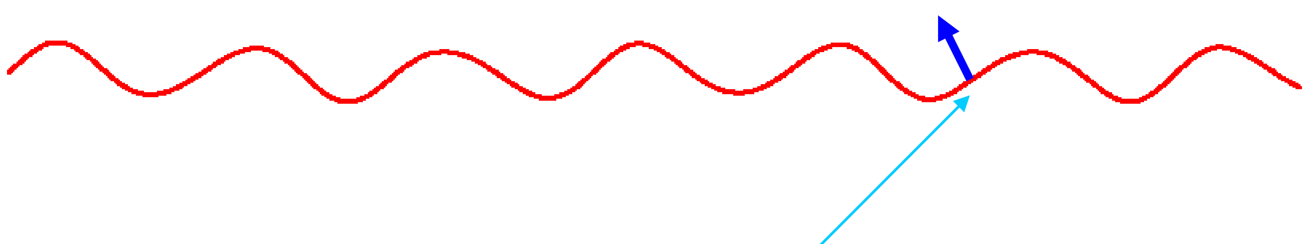
Particular free surface parametrization in 2D:

$$u = x \implies y = \eta(x,t)$$ - must be single-valued function of x

Kinematic condition: free surface moves with fluid:

$$V_n = \mathbf{n} \cdot (x_t, y_t) = \eta_t (1 + \eta_x^2)^{-1/2} = \partial_n \Phi_n|_{y=\eta} = \partial_n \Phi_s|_{y=\eta}$$

$$\partial_n \equiv \mathbf{n} \cdot \nabla$$



Unit normal vector:

$$\mathbf{n} = (-\nabla\eta, 1)\frac{1}{\sqrt{1 + (\nabla\eta)^2}}$$

Dynamic boundary condition:

$$P_\alpha = -\alpha \frac{\partial}{\partial x}[\eta_x (1+\eta_x^2)^{-1/2}]$$ - pressure jump at interface compare with zero pressure outside of fluid

Bernouilli equation for two fluid components[1]

$$\rho_n \left(\frac{\partial \Phi_n}{\partial t} + \frac{(\nabla \Phi_n)^2}{2} \right) + \rho_s \left(\frac{\partial \Phi_s}{\partial t} + \frac{(\nabla \Phi_s)^2}{2} \right) \Bigg|_{y=\eta} = \Gamma - P_\alpha - P_g,$$

$$P_g = \rho g \eta$$

$$\Gamma = \rho_n \rho_s V^2/(2\rho)$$ - Bernouilli constant

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[1]L.D. Landau and E.M. Lifshitz, Fluid mechanics, Pergamon (1989).

Laplace equations $\nabla^2\Phi_s = \nabla^2\Phi_n = 0$

+ kinematic and dynamic boundary conditions

$\Rightarrow$ a closed set of equations.

Hamiltonian formulation[1] similar to V.E. Zakharov (1968); E.A. Kuznetsov, M.D. Spector (1976); E.A. Kuznetsov, P.M. Lushnikov (1995)

$$\frac{\partial \Psi}{\partial t} = -\frac{\delta H}{\delta \eta},$$
$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \Psi},$$

- Canonical variables with the standard symplectic structure

where

$$\Psi_n = \Phi_n\big|_{y=\eta}, \quad \Psi_s = \Phi_s\big|_{y=\eta},$$

$$\Psi = \rho_n \Psi_n + \rho_s \Psi_s$$

[1]P.M. Lushnikov and N.M. Zubarev, JETP **156**, 711-721 (2019).

The Hamiltonian =kinetic energy+ potential energy, $H = K_n + K_s + U$

$$K_n = \frac{\rho_n}{2}\int \mathrm{d}x \int_{-\infty}^{\eta} \mathrm{d}y (\nabla\Phi_n)^2,$$

$$K_s = \frac{\rho_s}{2}\int \mathrm{d}x \int_{-\infty}^{\eta} \mathrm{d}y (\nabla\Phi_s)^2,$$

$$U = \frac{\rho_n + \rho_s}{2} g \int \mathrm{d}x\, \eta^2 + \alpha \int \mathrm{d}x \left(\sqrt{1+\eta_x^2} - 1\right)$$

potential energy in the gravitational field

surface tension energy

The Hamiltonian can be rewritten as a surface integral:

$$H = \frac{1}{2}\int \mathrm{d}x\, v_n \Psi \sqrt{1+\eta_x^2} + \frac{\rho_n + \rho_s}{2} g \int \mathrm{d}x\, \eta^2 + \alpha \int \mathrm{d}x \left(\sqrt{1+\eta_x^2} - 1\right)$$

Kelvin-Helmholtz instability dispersion relation [1] for linear perturbations $\propto e^{\mathbf{ik}\cdot\mathbf{r}_\perp - \mathbf{i}\omega t}$:

$$\omega_k^2 = \rho_s(\omega - \mathbf{V}_s\cdot\mathbf{k})^2 + \rho_n\left(\omega - \mathbf{V}_n\cdot\mathbf{k} + \mathrm{i}2\nu_n k^2\right)^2 + 4\rho_n\nu_n^2 k^3 m_n,$$

$$m_n = \left[k^2 - \mathrm{i}(\omega - \mathbf{V}_n\cdot\mathbf{k})/\nu_n\right]^{1/2}$$

$\omega_k^2 \equiv gk + \alpha k^3/\rho$ - dispersion law without average motion of Helium components

Comparison: dispersion law of the interface between two fluids of density ρ_n (upper fluid) and density ρ_s (lower fluid) without average motion [2]:

$$\omega_{k,classical}^2 = (\rho_s - \rho_n)gk/\rho + \alpha k^3/\rho$$

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[1] S.E. Korshunov, JETP Letters **75**, 496 (2002).
[2] L.D. Landau and E.M. Lifshitz, Fluid mechanics, Pergamon (1989).

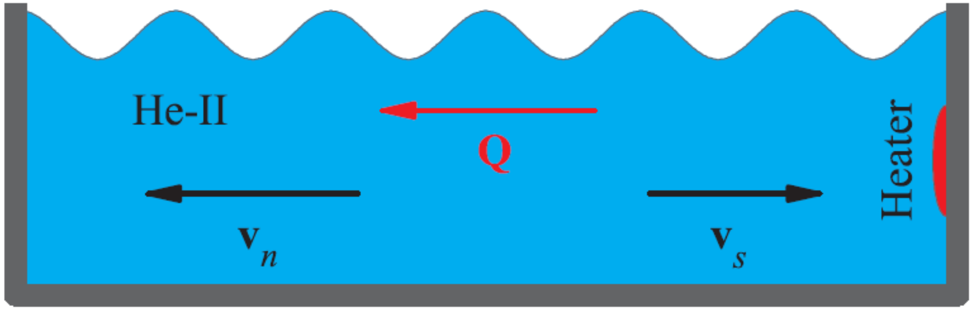
Fully nonlinear development of Kelvin-Helmholtz instability :

Define average velocity
and auxiliary potentials

$$\mathbf{v} = \frac{\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s}{\rho}$$

$$\Phi = (\rho_n \Phi_n + \rho_s \Phi_s)/\rho, \qquad \phi = \sqrt{\rho_n \rho_s}\,(\Phi_n - \Phi_s)/\rho.$$

$$\nabla \Phi = \mathbf{v}$$



$$\Rightarrow \begin{cases} \nabla^2 \Phi = 0, \qquad \nabla^2 \phi = 0, \\ \dfrac{\partial \Phi}{\partial t} + \dfrac{(\nabla \Phi)^2}{2} + \dfrac{(\nabla \phi)^2}{2} = \dfrac{\Gamma - P_\alpha + P_g}{\rho} & y = \eta, \\ \eta_t (1 + \eta_x^2)^{-1/2} = \partial_n \Phi & \text{at} \quad y = \eta, \\ \partial_n \phi = 0 & \text{at} \quad y = \eta, \\ \Phi \to 0 & |x| \to \infty \quad \text{or} \quad y \to -\infty, \\ \phi \to -Vx\sqrt{\rho_n \rho_s}/\rho & |x| \to \infty \quad \text{or} \quad y \to -\infty. \end{cases}$$

Replacing ϕ by the harmonic conjugate function ψ
($\phi_x = \psi_y$ and $\phi_y = -\psi_x$)

$\Rightarrow$

$$
\begin{cases}
\nabla^2\Phi = 0, \qquad \nabla^2\psi = 0, \\
\dfrac{\partial\Phi}{\partial t} + \dfrac{(\nabla\Phi)^2}{2} + \dfrac{(\nabla\psi)^2}{2} = \dfrac{c^2}{2} - \dfrac{P_\alpha + P_g}{\rho} \quad \textbf{at } y = \eta, \\
\eta_t(1+\eta_x^2)^{-1/2} = \partial_n\Phi \quad \textbf{at } y = \eta, \\
\partial_\tau\psi = 0 - \textbf{tangential derivative at } y = \eta, \\
\Phi \to 0 \quad \textbf{at } |x| \to \infty \textbf{ or } y \to -\infty, \\
\psi \to -cy \quad \textbf{for } |x| \to \infty \textbf{ and } y \to -\infty, \qquad \textbf{where } c = \sqrt{2\Gamma/\rho}.
\end{cases}
$$

Define a pair of new harmonic potentials $F^{(\pm)} = (\Phi \pm \psi \pm cy)/2.$

Then for zero gravity and surface tension[1]**:**

$$\begin{cases} \nabla^2 F^{(+)} = 0, \\ F_t^{(+)} - cF_y^{(+)} + (\nabla F^{(+)})^2 = 0 \text{ at } y = \eta, \\ F^{(+)} \to 0 \text{ at } |x| \to \infty \text{ and } y \to -\infty, \end{cases} \qquad \begin{cases} \nabla^2 F^{(-)} = 0, \\ F_t^{(-)} + cF_y^{(-)} + (\nabla F^{(-)})^2 = 0 \text{ at } y = \eta, \\ F^{(-)} \to 0 \text{ at } |x| \to \infty \text{ and } y \to -\infty, \end{cases}$$

$$c\eta = F^{(+)} - F^{(-)}\Big|_{y=\eta}.$$

These equations are fully decoupled if either $F^{(+)} = 0,$ **or** $F^{(-)} = 0.$

Physical interpretation of these reductions: correspond to either stable or unstable Branches of the liner dispersion relation $\omega^{(\pm)} = \pm ick,$

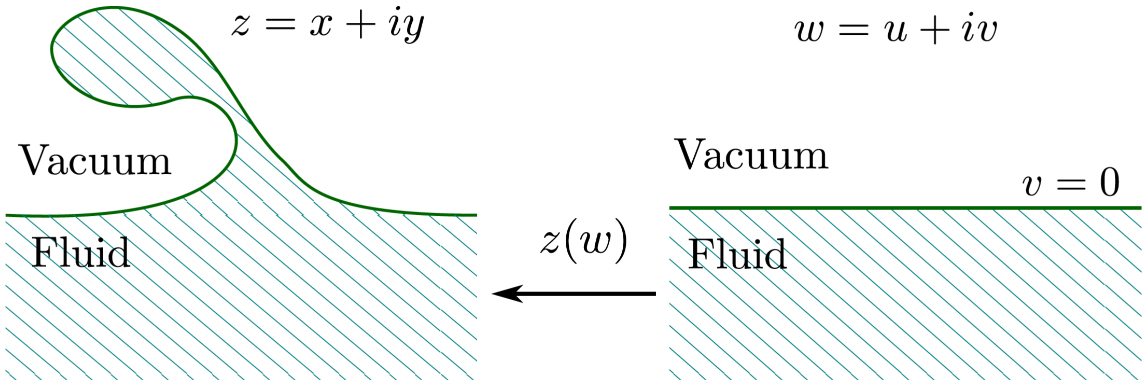
$$\eta = a^{(+)} \exp(ikx - i\omega^{(+)}t) + a^{(-)} \exp(ikx - i\omega^{(-)}t),$$

$$F^{(\pm)} = \pm c a^{(\pm)} \exp(ikx + ky - i\omega^{(\pm)}t).$$

---

[1] P.M. Lushnikov and N.M. Zubarev, PRL, **120**, 204504 (2018).

Using conformal variables for linearly unstable reduction $F^{(-)} = 0.$



$\Rightarrow$ **Laplace growth equation**[1] $\boxed{\mathrm{Im}\left(\bar{G}_t G_u\right) = c.}$

for $G(u,t) = z(u,t) - ict$

Laplace growth equation has infinite number of integral of motion and exact solutions[2-7]. It is also integrable in a sense of the close relation with the dispersionless limit of the integrable Toda hierarchy[8].

[1]P.M. Lushnikov and N.M. Zubarev, PRL, 120, 204504 (2018).
[2]P.Ya. Polubarinova-Kochina, Dokl. Akad. Nauk SSSR 47, 254 (1945).
[3]L.A. Galin, Dokl. Akad. Nauk SSSR 47, 246 (1945).
[4]B.I. Shraiman and D. Bensimon, Phys. Rev. A 30, 2840 (1984).
[5]S.D. Howison, SIAM J. Appl. Math. 46, 20 (1986).
[6]D. Bensimon, L.P. Kadanoff, S. Liang, B.I. Shraiman, and C. Tang, Rev. Mod. Phys. 58, 977 (1986).
[7]M. Mineev-Weinstein, P. B. Wiegmann, and A. Zabrodin, Phys. Rev. Lett. 84, 5106 (2000).
[8] I. Krichever, M. Mineev-Weinstein, P. Wiegmannd, A. Zabrodin, Phys. D 198, 1-28 (2004).

Localized solutions of Laplace growth equation

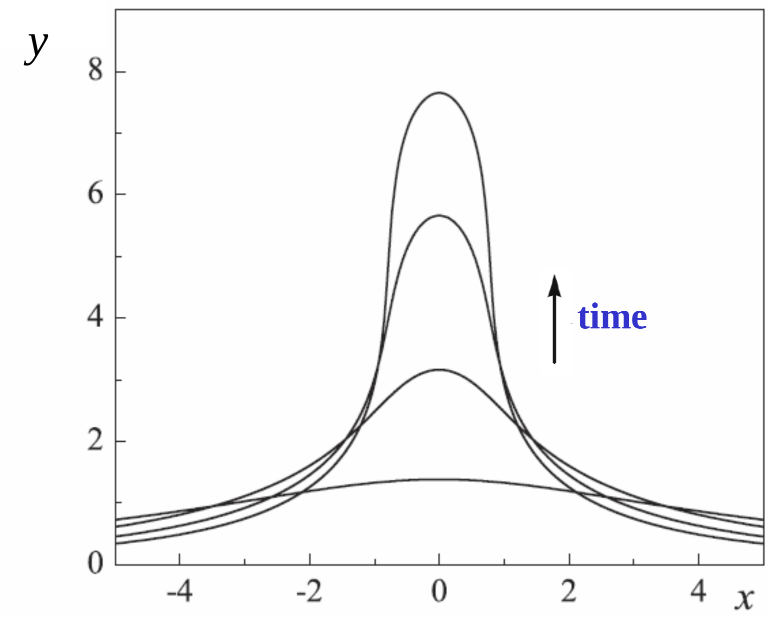
$$\mathrm{Im}\left(\bar{G}_t G_u\right) = c.$$

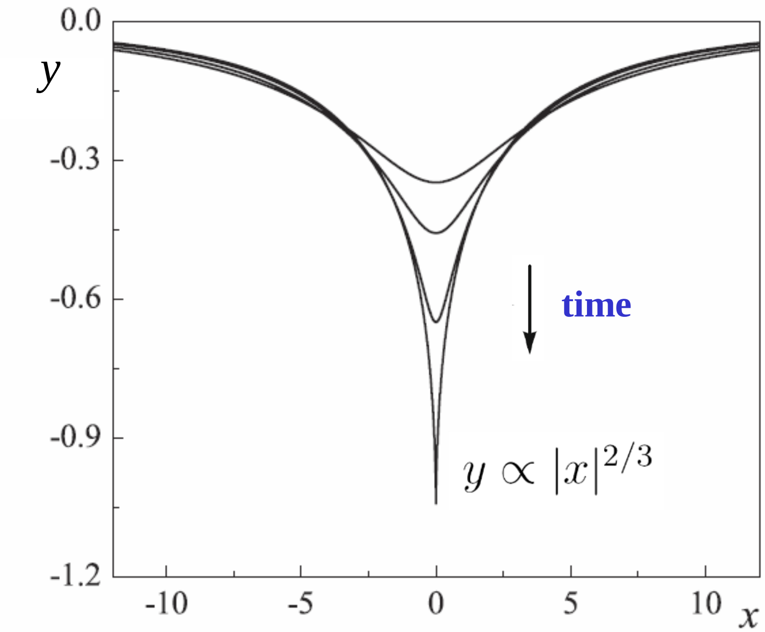
Multi-finger and multi-cusp set of solutions:

$$G(u,t) = u - ict - i\sum_{n=0}^{N} a_n \ln(u - w_n(t)), \qquad a_0 = -\sum_{n=1}^{N} a_n,$$

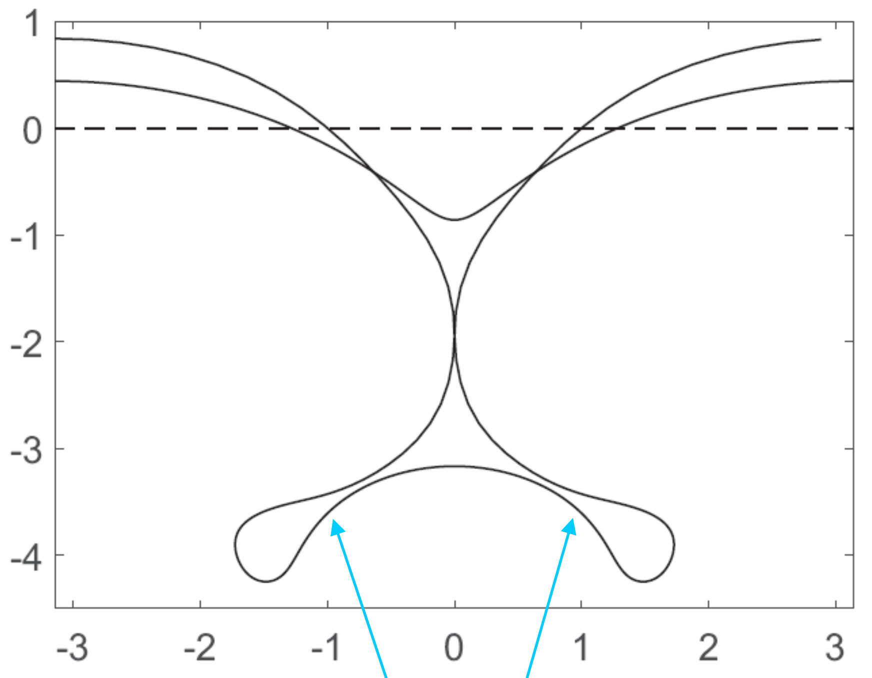
$$w_n + ict + i\sum_{m=0}^{N} \bar{a}_m \ln(w_n - \bar{w}_m) = C_n, \qquad n = 0, 1, 2, \ldots, N.$$

Spatial profiles of solutions in physical variables for $N = 1, \quad a_1 = \pm 1.$





Regularization of cusp by the finite surface tension



Crapper-like solutions

[1]E.A. Kochurin, P.M. Lushnikov and N.M. Zubarev (2019).

Hamiltonian formalism for the reduced dynamical equations[1]

$$\mathbf{Q}_t = \hat{R}\frac{\delta H}{\delta \mathbf{Q}}, \quad \mathbf{Q} \equiv \begin{pmatrix} y \\ \psi \end{pmatrix}$$

Modified Hamiltonian:

$$H = H_{Eul} + \tilde{H},$$

$$H_{Eul} = -\frac{1}{2}\int\limits_{-\infty}^{\infty} \psi\hat{\mathcal{H}}\psi_u \mathrm{d}u + \frac{g}{2}\int\limits_{-\infty}^{\infty} y^2\,(1-\hat{\mathcal{H}}y_u)\mathrm{d}u + \alpha\int\limits_{-\infty}^{\infty}\left(\sqrt{(1-\hat{\mathcal{H}}y_u)^2+y_u^2}-1+\hat{\mathcal{H}}y_u\right)\mathrm{d}u$$

$$= \int\limits_{-\infty}^{\infty} du\left[\frac{\mathrm{i}}{8}(\Pi_u\bar{\Pi}-\Pi\bar{\Pi}_u) - \frac{g}{16}(z-\bar{z})^2(z_u+\bar{z}_u) + \alpha\left(\sqrt{z_u\bar{z}_u}-\frac{z_u+\bar{z}_u}{2}\right)\right]$$

- standard Hamiltonian for a single fluid

$$\tilde{H} = \frac{\mathrm{i}}{8}\int (z_u+\bar{z}_u-2)(z-\bar{z})\mathrm{d}u = \frac{1}{2}\int y\hat{\mathcal{H}}y_u \mathrm{d}u$$

- additional term in the Hamiltonian

For zero surface tension and gravity as well as the reduction $\Pi_u = \mathrm{i}c(z_u-1)$ we obtain Laplace growth equation

$$\mathrm{Im}\left(\bar{G}_t G_u\right) = c \qquad G(u,t) = z(u,t) - ict$$

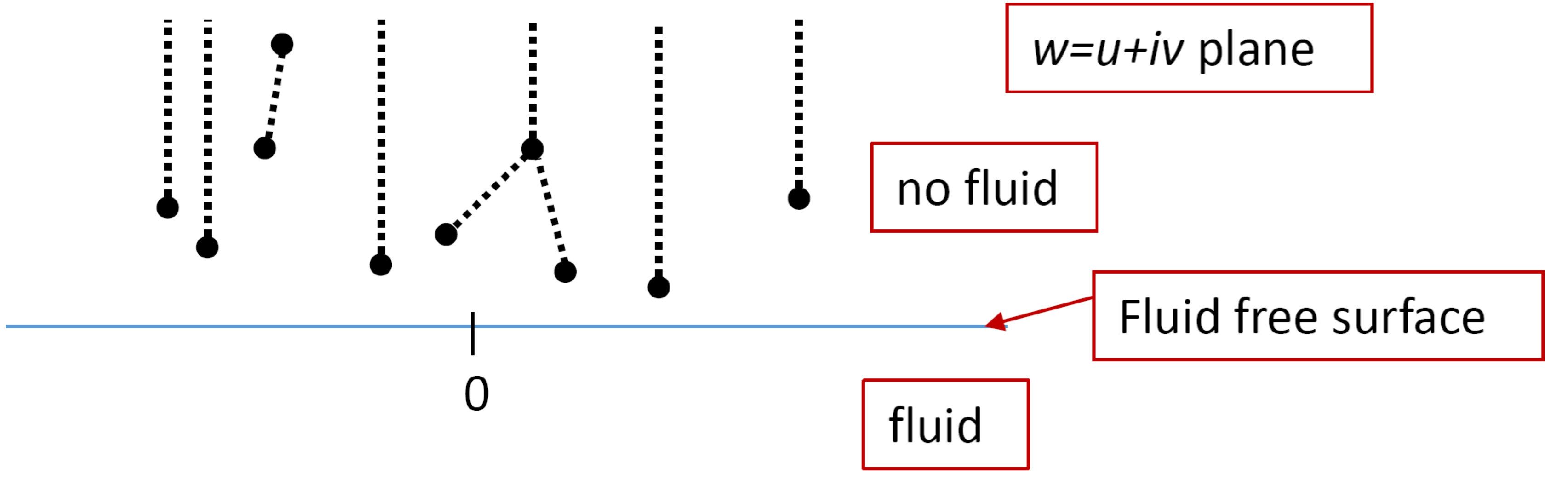
---

[1]A.I. Dyachenko, P. M. Lushnikov and V. E. Zakharov, Non-Canonical Hamiltonian Structure and Poisson Bracket for 2D Hydrodynamics with Free Surface, J. of Fluid Mech. **869**, 526-552 (2019).

Returning to Euler equations with free surface for a single fluid:

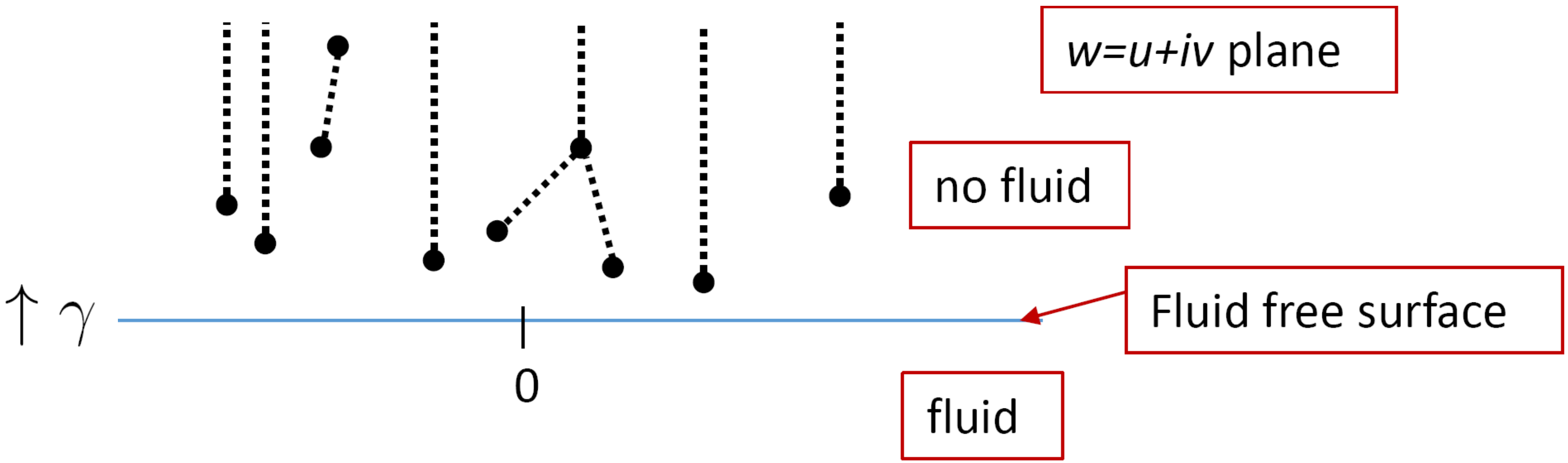
Water waves even in **2D** are not integrable in a sense of inverse scattering transform with time-independent spectral parameter (fourth order matrix element is zero while 5th order is not zero on resonance surfaces) [1].

Instead our general program is to fully describe **2D** hydrodynamics of idea fluid with free surface by the dynamics of complex singularities outside of fluid and find the Infinite set of the integrals of motion.

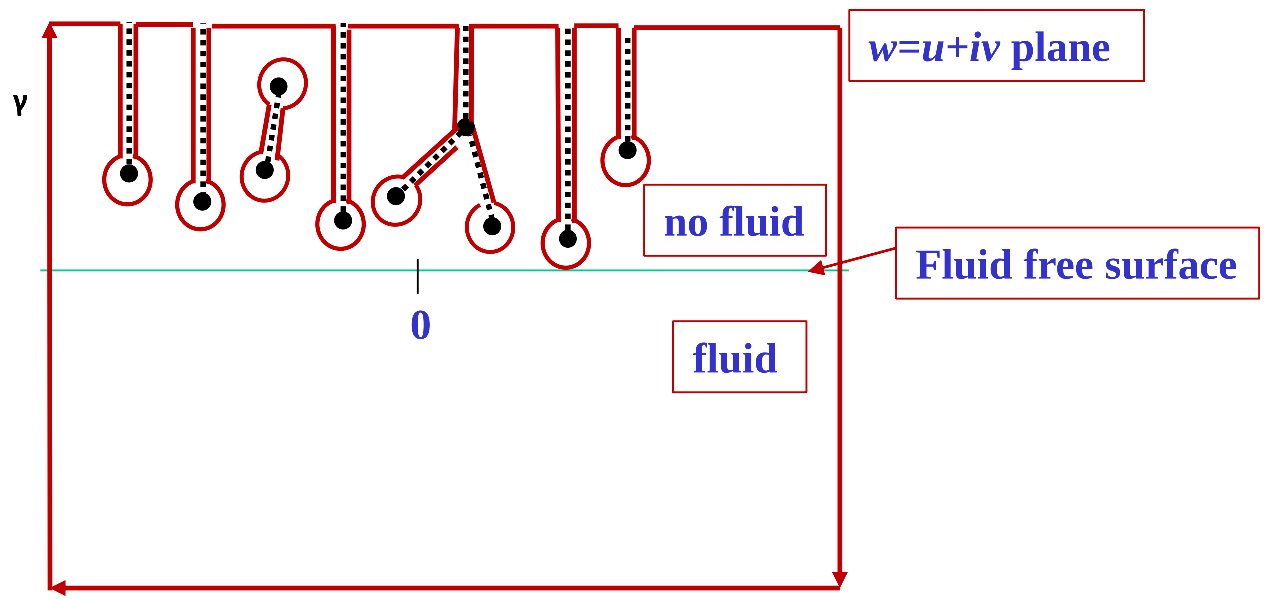


[1] A.I. Dyachenko, Y.V. Lvov and V.E. Zakharov, Phys. D **87**, 233-261 (1995).

Cauchy's integral formula $f(w) = (2\pi \mathrm{i})^{-1} \oint_\gamma \frac{f(w')dw'}{w'-w}$



Moving integration contour γ:



$$z(w) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{\rho(w')dw'}{w-w'}$$

$\rho(w')$ - jump of $z(w)$ at each branch cut

Numerics: rational approximation of branch cuts

$$z(w) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{\rho(w')dw'}{w - w'} \simeq \frac{1}{2\pi \mathrm{i}} \sum_j \frac{\rho(w_j)\delta w_j}{w - w_j}$$

Straightforward rational approximation – Pade approximation is extremely ill-posed.

But use least-square-based rational approximation[1,2] to avoid ill-posedness as well develop a series of additional conformal maps Which are compatible with the hydrodynamics and move complex singularities away from real axis[2,3]

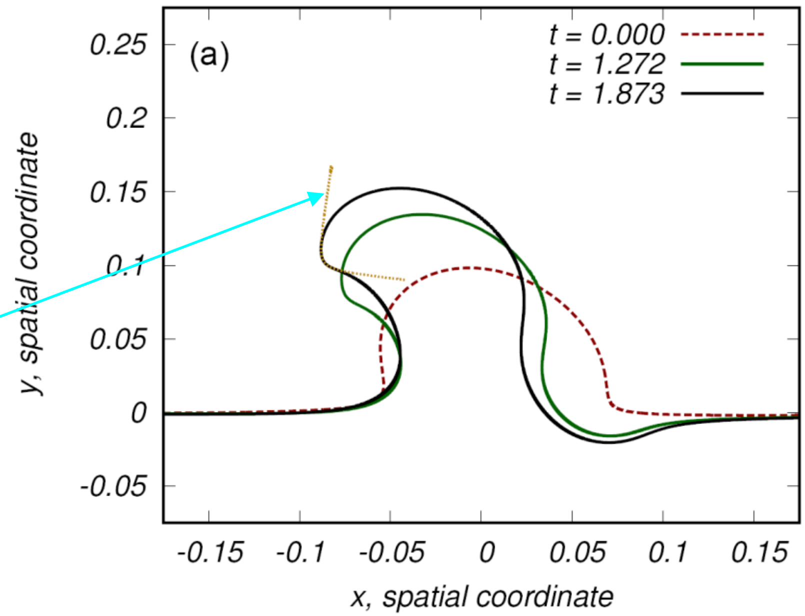
---

[1]B. Alpert, L. Greengard, and T. Hagstrom, SIAM J. Num. Anal. **37**, 1138– 1164 (2000).
[2]S.A. Dyachenko, P.M. Lushnikov, and A.O. Korotkevich, Stud. Appl. Math., **137**, 419-472 (2016).
[3]P. M. Lushnikov, S.A. Dyachenko and D.A. Silantyev, Proc. Roy. Soc. A, **473,** 20170198 (2017).

Example: Motion of two poles coupled with moving branch cuts

Shape of surface at different times:

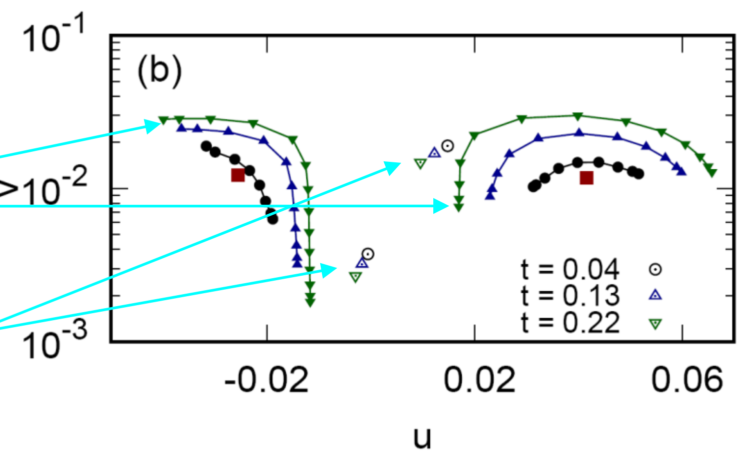
Fit to square root singularity



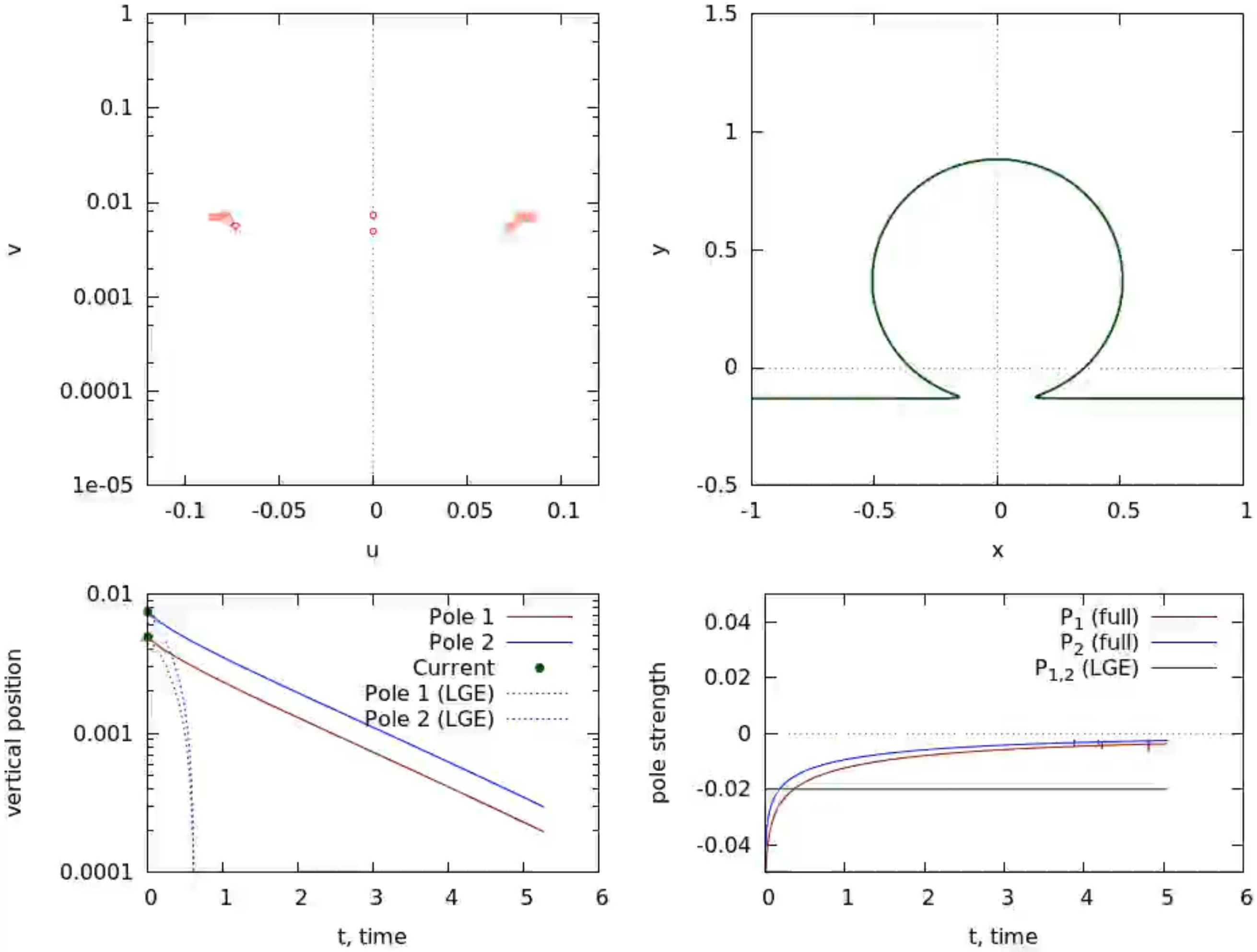
Complex plane:

A pair of branch cuts

A pair of poles



Example: Motion of two poles along imaginary axis



Analytical solutions:
Pole solutions coupled with branch cuts with $\boldsymbol{\alpha = 0}$ [1]

$$z_w = \sum_{k=1}^{N} \frac{c_1^{(k)}(t)}{w - a_k(t)} + (z_w)_{regular}, \qquad \Pi_w = \sum_{k=1}^{N} \frac{W_{-1}^{(k)}(t)}{w - a_k(t)} + (\Pi_w)_{regular},$$

$$c_1^{(k)}(t) = const = c^{(k)}, \quad W_{-1}^{(k)}(t) = -\mathrm{i}(e_1^{(k)} - gt)c_1^{(k)}, \quad e_1^{(k)} = const$$

Residue of z_w is the complex constant of motion

Residue of $\boldsymbol{\Pi_w}$ is the complex constant of motion $g=0$

$(z_w)_{regular}, \ (\Pi_w)_{regular}$ - analytic functions at $w = a_k$ but generally include branch points at other points $w_0 \in \mathbb{C}^+$ with often $(z_w)_{regular}, \ (\Pi_w)_{regular} \propto \sqrt{w - w_0}$

---

[1] A.I. Dyachenko, S. A. Dyachenko, P. M. Lushnikov and V. E. Zakharov, Dynamics of Poles in 2D Hydrodyn. with Free Surface: New Constants of Motion, J. of Fluid Mech. **874**, 891-925 (2019).

Number of independent integrals of motion

$$c^{(k)}(t) = const = c^{(k)}, \quad W_{-1}^{(k)}(t) = -\mathrm{i}(e^{(k)} - gt)c^{(k)}, \quad e^{(k)} = const$$

2N real integrals

2N-1 real integrals

$$\frac{W_{-1}^{(k)}(t)}{c_1^{(k)}} - \frac{W_{-1}^{(N)}(t)}{c_1^{(N)}}, \quad k = 1, 2, \ldots, N-1$$

for $\boldsymbol{g} \neq \mathbf{0}$

or *2N* integrals for $\boldsymbol{g} = \mathbf{0}$

Total number of real integrals of motion:

4N-1 for $\boldsymbol{g} \neq \mathbf{0}$

4N for $\boldsymbol{g} = \mathbf{0}$

Towards possible integrability of Euler equations with free surface

Residues of z_w provide an infinite number of commuting integrals of motion

$$\{c_1^{(n)}, c_1^{(k)}\} = 0 \qquad n, k = 1, \ldots, N.$$

as $$\frac{\delta c_1^{(n)}}{\delta \psi} = 0, \;\; n = 1, \ldots, N$$ and

$$\{F, G\} = \sum_{i,j=1}^{2} \int_{-\infty}^{\infty} \mathrm{d}u \left(\frac{\delta F}{\delta Q_i} \hat{R}_{ij} \frac{\delta G}{\delta Q_j} \right) = \int_{-\infty}^{\infty} \mathrm{d}u \left(\frac{\delta F}{\delta y} \hat{R}_{12} \frac{\delta G}{\delta \psi} + \frac{\delta F}{\delta \psi} \hat{R}_{21} \frac{\delta G}{\delta y} + \frac{\delta F}{\delta \psi} \hat{R}_{22} \frac{\delta G}{\delta \psi} \right)$$

Commuting integrals of motion are precursor of the Hamiltonian integrability[1-3].

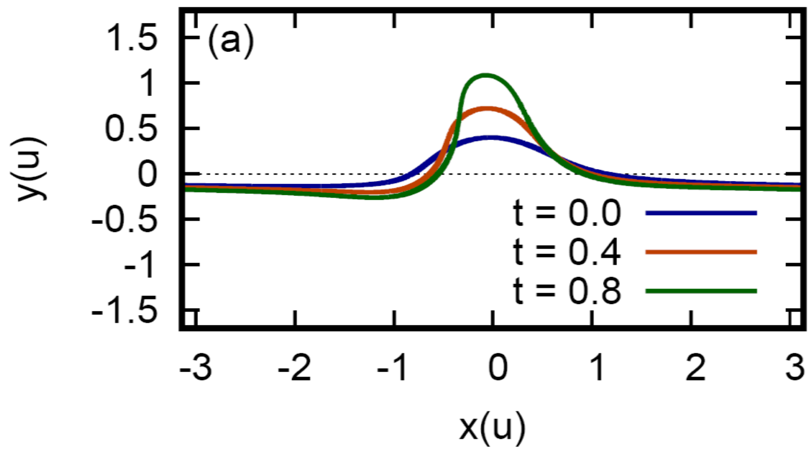
Strong arguments in favor of the Hamiltonian integrability!

---

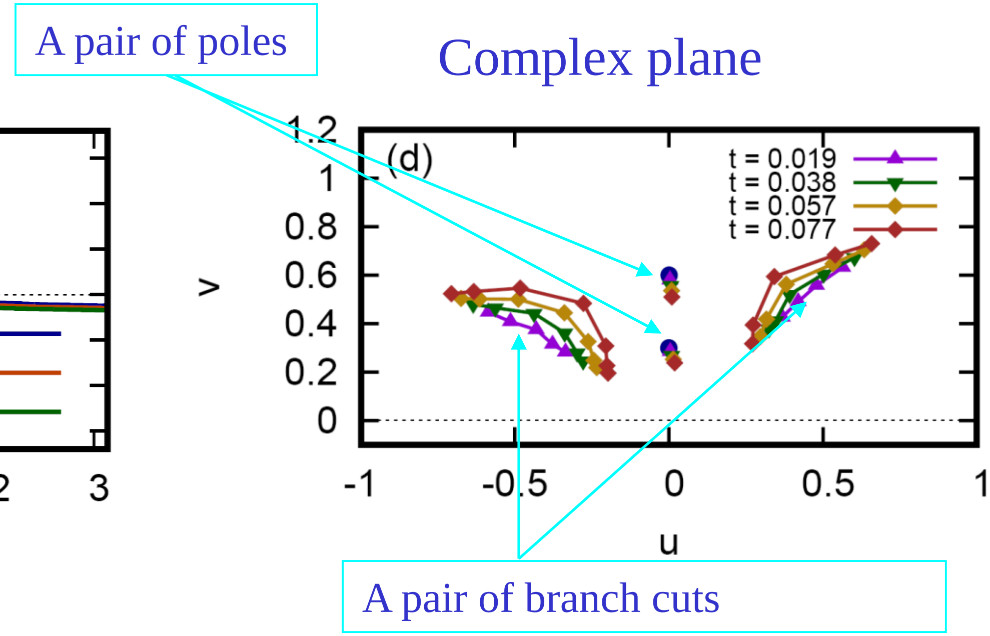
[1]C.S. Gardner and J.M. Greene, M.D. Kruskal and R.M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).
[2]V.E. Zakharov and L.D. Faddeev, Funct. Anal. Appl. **5**, 280 (1971).
[3]V.E. Zakharov and A.B. Shabat, Sov. Phys. JETP. **34**, 62 (1972).

Integrals of motion in numerics with rational initial conditions

Surface profiles

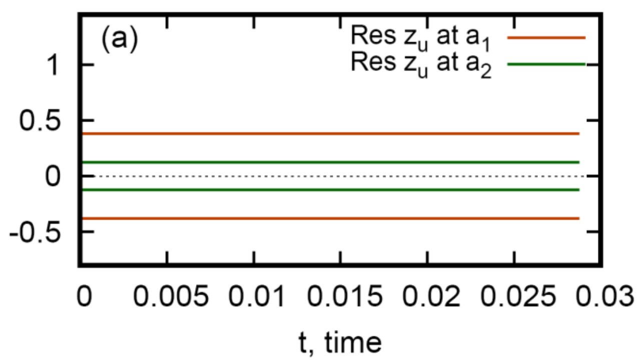


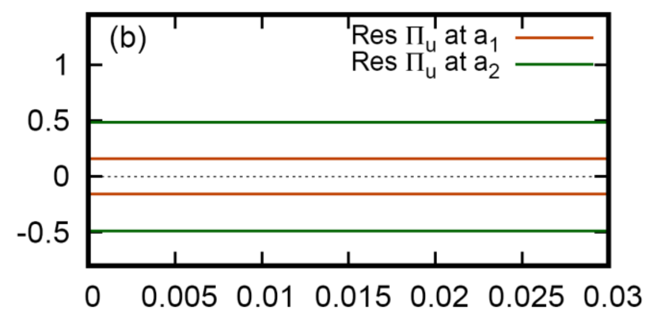
Complex plane



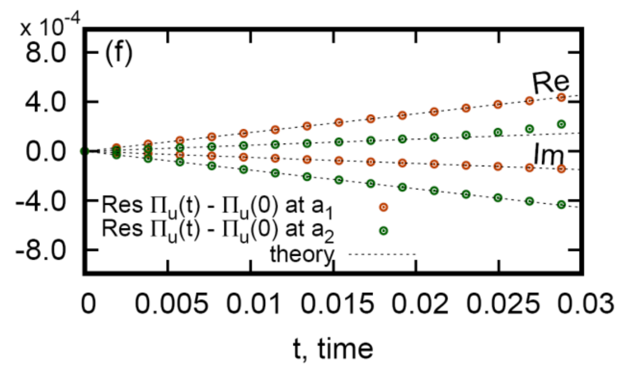
Time dependence of residues

$g = 0$





$g \neq 0$



New constants of motion for nonzero surface tension and second order zero of *R* for Dyachenko variables[1]

$$R = \frac{1}{z_w},$$

$$V = i\Pi_z = iR\Pi_w$$

$$\frac{\partial R}{\partial t} = \mathrm{i}\left(UR_u - RU_u\right), \quad U = \hat{P}^{-}(R\bar{V} + \bar{R}V),$$

$$\frac{\partial V}{\partial t} = \mathrm{i}\left[UV_u - R\hat{P}\frac{\partial}{\partial u}(V\bar{V})\right] + g(R-1) - \alpha R\hat{P}^{-}\frac{\partial}{\partial u}\left(\frac{R_u\sqrt{\bar{R}}}{\sqrt{R}} - \frac{\bar{R}_u\sqrt{R}}{\sqrt{\bar{R}}}\right)$$

$$\hat{P}^{\pm}q = \frac{1}{2}(1 \mp \mathrm{i}\hat{H})q$$

- Projector operators to functions analytic in upper (lower) complex half-planes

Second order zero – compatible with surface tension term:

$$R = R_2(w-a)^2 + \ldots, \quad R_2 \neq 0,$$

$$V = V_0 + V_1(w-a) + V_2(w-a)^2 + \ldots,$$

$$\Rightarrow \quad a_t = -\mathrm{i}U_0$$

$$\frac{R_2}{V_1} = \mathrm{const} \equiv c_2^{(1)}$$

$$V_0(t) = -gt + e^{(1)}$$

Complex constants of motion

---

[1]A.I. Dyachenko, Doklady Math, **53**, 115 (2001).

Returning to variables of $\mathbf{z_w}$ and $\boldsymbol{\Pi_w}$

$$z_w = \frac{1}{R_2(w-a)^2} - \frac{R_3}{R_2^2(w-a)} + O\left((w-a)^0\right),$$

$$\Pi_w = \frac{-\mathrm{i}V_0}{R_2(w-a)^2} + \frac{\mathrm{i}(R_3V_0 - R_2V_1)}{R_2^2(w-a)} + O\left((w-a)^0\right)$$

$$\underset{w=a}{Res}(z_w) \equiv c_2^{(1)} = const \qquad \underset{w=a}{Res}(\Pi_w) + \mathrm{i}\underset{w=a}{Res}(z_w)V_0 = const$$

$$V_0(t) = -gt + e^{(1)}$$

Total number of real integrals of motion:

6N-1 for $\boldsymbol{g \neq 0}$

6N for $\boldsymbol{g = 0}$

New constants of motion for nonzero surface tension and higher order zeros of for *m*th order zero of *R* (*m* must be even for nonzero Surface tension)

$$R = R_m (w-a)^m + \dots, \ R_m \neq 0,$$

$$V = V_0 + V_1(w-a) + V_2(w-a)^2 + \dots .$$

$$z_w = \frac{1}{R_m(w-a)^m} + O\left((w-a)^{-m+1}\right),$$

$$\Pi_w = \frac{-\mathrm{i}V_0}{R_m(w-a)^m} + O\left((w-a)^{-m+1}\right).$$

$$a_t = -\mathrm{i}U_0$$

$$\Rightarrow \quad \frac{R_m}{V_1^{m-1}} = \text{const} \equiv c_m^{(1)}, \ m > 1$$

$$V_0(t) = -gt + e^{(1)}$$

Complex constants of motion

Total number of real integrals of motion:

8N-1 for $\boldsymbol{g} \neq \mathbf{0}$ *8N* for $\boldsymbol{g} = \mathbf{0}$ for *m*>2

Local analysis: persistence of branch cuts

$$V = V_0 + V_\alpha (w-a)^\alpha + \ldots,$$
$$R = R_0 + R_\alpha (w-a)^\alpha + \ldots,$$
$$U = U_0 + U_\alpha (w-a)^\alpha + \ldots,$$
$$B = B_0 + B_\alpha (w-a)^\alpha + \ldots,$$

$$0 < Re(\alpha) < 1$$

Generic case: $\alpha = \frac{1}{2}$

any rational α is allowed

Origin of a pair of branch points from $z_w = 0$ was also found in Refs. [1-3] and square roots singularities were studied in Refs. [4-10]

---

[1] S. Tanveer, Proc. R. Soc. Lond. A **435**, 137-158 (1991).

[2] S. Tanveer, Proc. R. Soc. Lond. A **441**, 501-525 (1993).

[3] E.A. Kuznetsov, M.D. Spector, and V.E. Zakharov, Physics Letters A 182, 387-393 (1993).

[4] Moore, D. W. Proc. R. Soc. Lond. A **365**, 105 (1979).

[5] D. I. Meiron, G. R. Baker, and S. A. Orszag, J. Fluid Mech. **114**, 283 (1982).

[6] R. Krasny, J. Fluid Mech. **167**, 65 (1986).

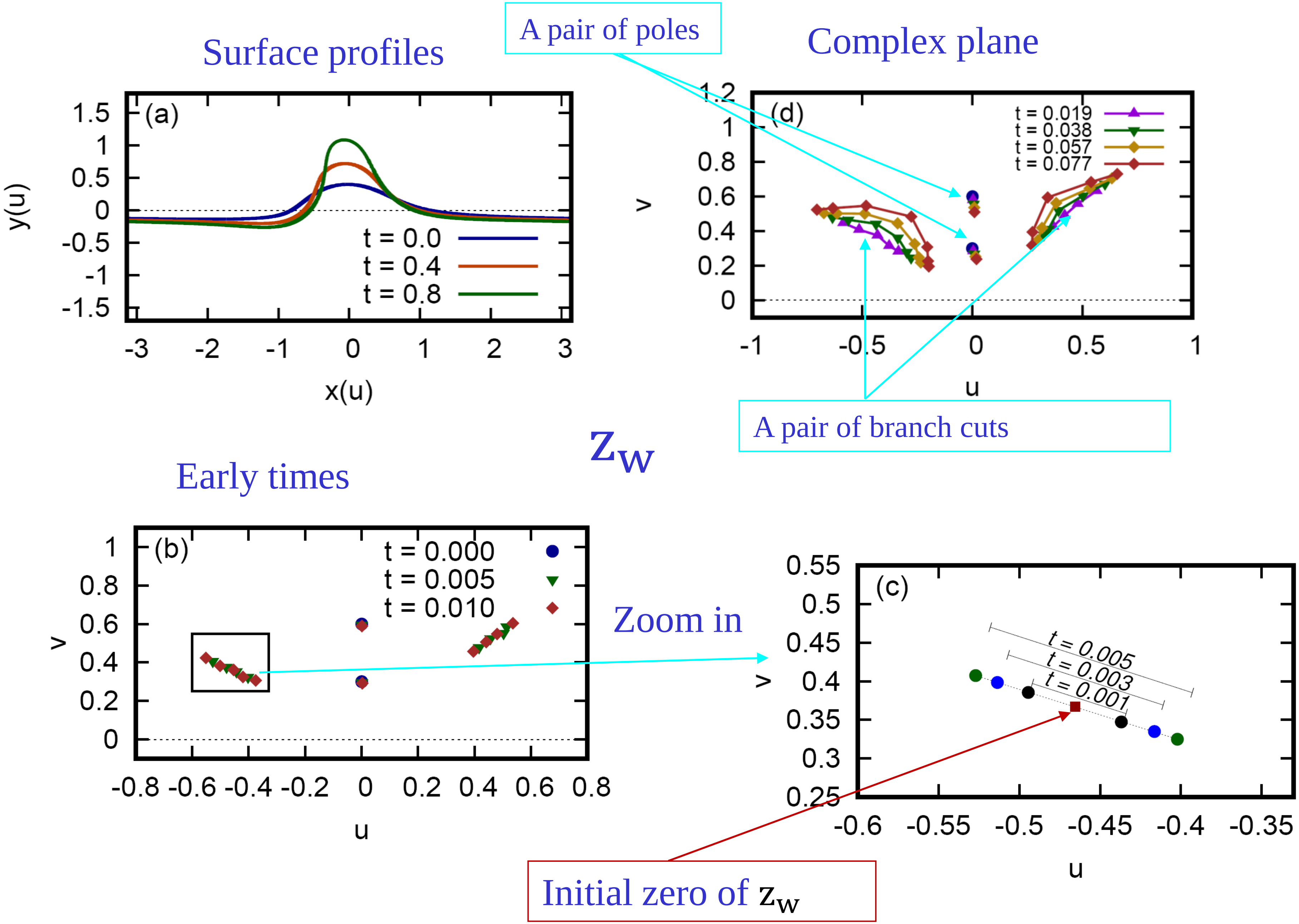
[7] R. Caflisch and O. Orellana, SIAM J. Math. Anal. **20**, 293 (1989).

[8] M. Shelley, J. Fluid Mech. **244**, 493 (1992).

[9] R.E. Caflisch, G. Baker, and M. Siegel. J. Fluid Mech. **252**, 51-78 (1993).

[10] S. J. Cowley, G. R. Baker, and S. Tanveer, J. Fluid Mech. **378**, 233–267 (1999).

Example: formation of branch cuts from rational initial conditions

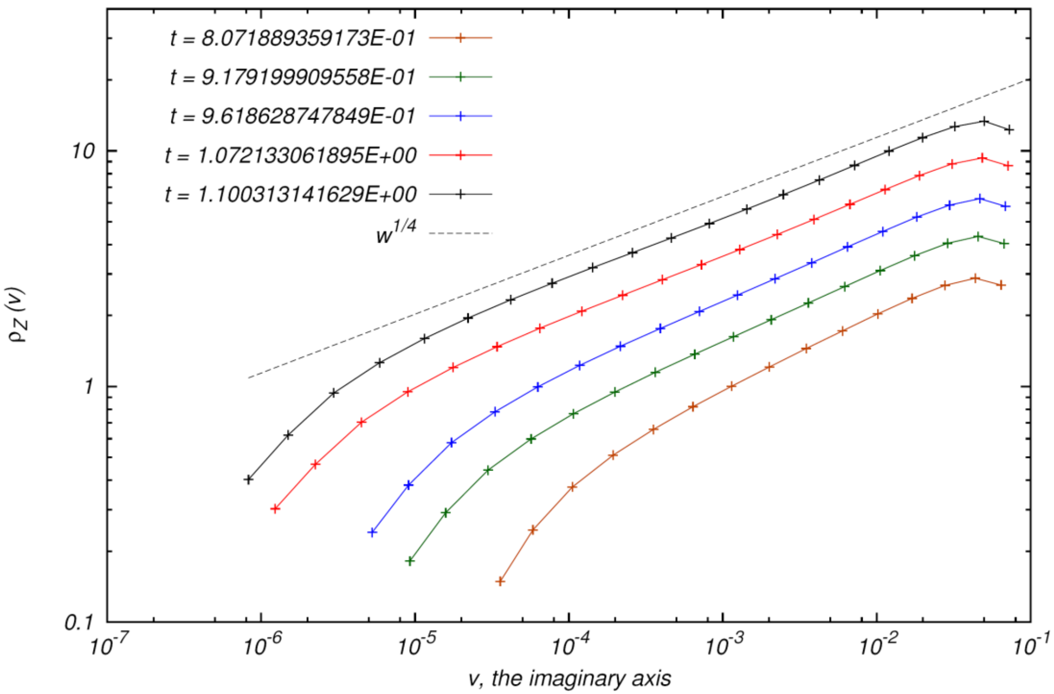


Global analysis: conjecture that generically *R* and *V* only have branch cuts and poles so they have the same Riemann surface.

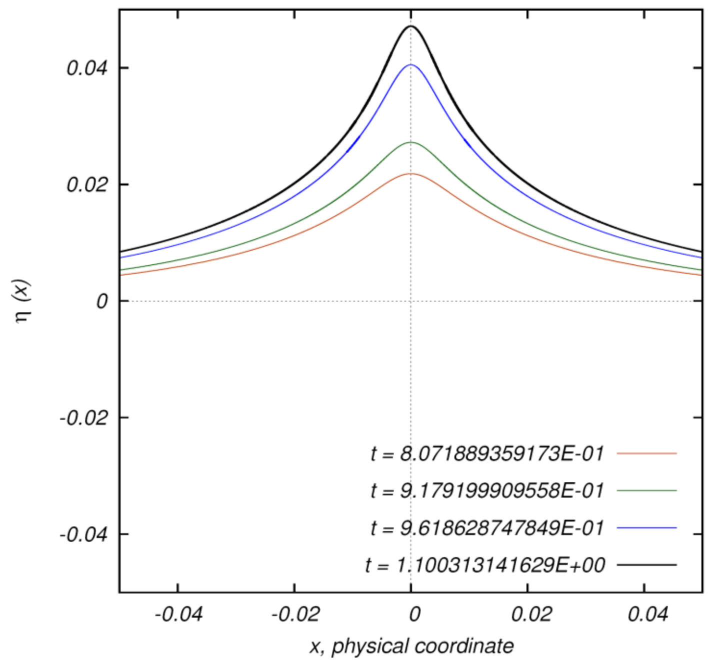
Have to study other sheets of Riemann surface. It suggests to use nested square root technique qualitatively similar to the infinite number of sheets in Stokes wave [1]

Example: jump at the branch cur for moving jet.

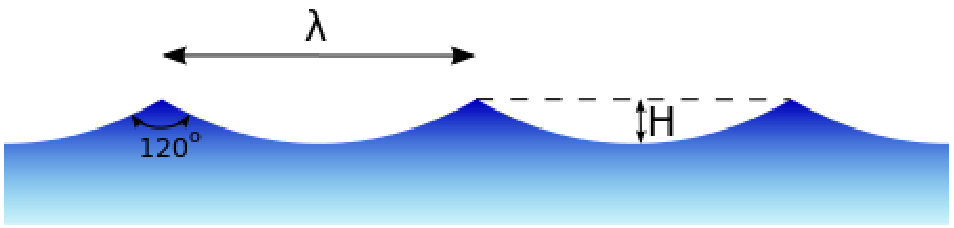
Jump at the branch cut: ¼ power is the indication of infinite number of sheets of Riemann surface



Spatial profiles at different times



Recovering 2/3 power law of limiting Stokes wave from ½ power law singularities in the limit of branch point approaching the real line[1] $\chi_c \to 0$



$$z \propto \left[(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}\right]\sqrt{\alpha_1\chi_c^{1/4} + \sqrt{(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}$$

$$\times\sqrt{\alpha_3\chi_c^{1/16} + \sqrt{\alpha_2\chi_c^{1/8} + \sqrt{\alpha_1\chi_c^{1/4} + \sqrt{(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}}} \times \cdots$$

$$\zeta = \tan\frac{w}{2}$$

$$\zeta \gg \chi_c \quad \Rightarrow \quad \boxed{z \propto \zeta^{1/2+1/8+1/32+\ldots} = \zeta^{2/3}}$$

---

[1] P. M. Lushnikov, Journal of Fluid Mechanics, **800**, 557-594 (2016)

Expression under the most inner square root:

$$g(\zeta) \equiv (\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}$$

Two branches at $\boldsymbol{\zeta = -\mathrm{i}\chi_c}$:

$$g_+(\zeta) = 2(-2\mathrm{i}\chi_c)^{1/2} + \frac{\zeta + \mathrm{i}\chi_c}{2(-2\mathrm{i}\chi_c)^{1/2}} + O(\zeta + \mathrm{i}\chi_c)^2$$

- no singularity of $\sqrt{g(\zeta)}$

$$g_-(\zeta) = -\frac{\zeta + \mathrm{i}\chi_c}{2(-2\mathrm{i}\chi_c)^{1/2}} + O(\zeta + \mathrm{i}\chi_c)^2$$

- singularity of $\sqrt{g(\zeta)}$ at $\boldsymbol{\zeta = -\mathrm{i}\chi_c}$

More details on solution

$$z = \mathrm{i}\frac{c^2}{2} + c_1\chi_c^{1/6}\sqrt{\zeta - \mathrm{i}\chi_c}$$
$$+ \frac{(3c)^{2/3}}{2}e^{-\mathrm{i}\pi/6}\left[(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}\right]\sqrt{\alpha_1\chi_c^{1/4} + \sqrt{(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}$$
$$\times\sqrt{\alpha_3\chi_c^{1/16} + \sqrt{\alpha_2\chi_c^{1/8} + \sqrt{\alpha_1\chi_c^{1/4} + \sqrt{(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}}}$$
$$\times\sqrt{\alpha_{2n+1}\chi_c^{1/2^{2n+2}} + \sqrt{\alpha_{2n}\chi_c^{1/2^{2n+1}} + \sqrt{\ldots + \sqrt{\alpha_1\chi_c^{1/4} + \sqrt{(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}}}}$$
$$\times\cdots + \frac{(3c)^{2/3}}{2}e^{-\mathrm{i}\pi/6}\left[(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}\right]\sqrt{\tilde{\alpha}_1\chi_c^{1/4} + \sqrt{(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}$$
$$\times\sqrt{\tilde{\alpha}_3\chi_c^{1/16} + \sqrt{\tilde{\alpha}_2\chi_c^{1/8} + \sqrt{\tilde{\alpha}_1\chi_c^{1/4} + \sqrt{(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}}}$$
$$\times\sqrt{\tilde{\alpha}_{2n+1}\chi_c^{1/2^{2n+2}} + \sqrt{\tilde{\alpha}_{2n}\chi_c^{1/2^{2n+1}} + \sqrt{\ldots + \sqrt{\tilde{\alpha}_1\chi_c^{1/4} + \sqrt{(\zeta - \mathrm{i}\chi_c)^{1/2} + (-2\mathrm{i}\chi_c)^{1/2}}}}}}$$
$$\times \cdots + h.o.t.$$

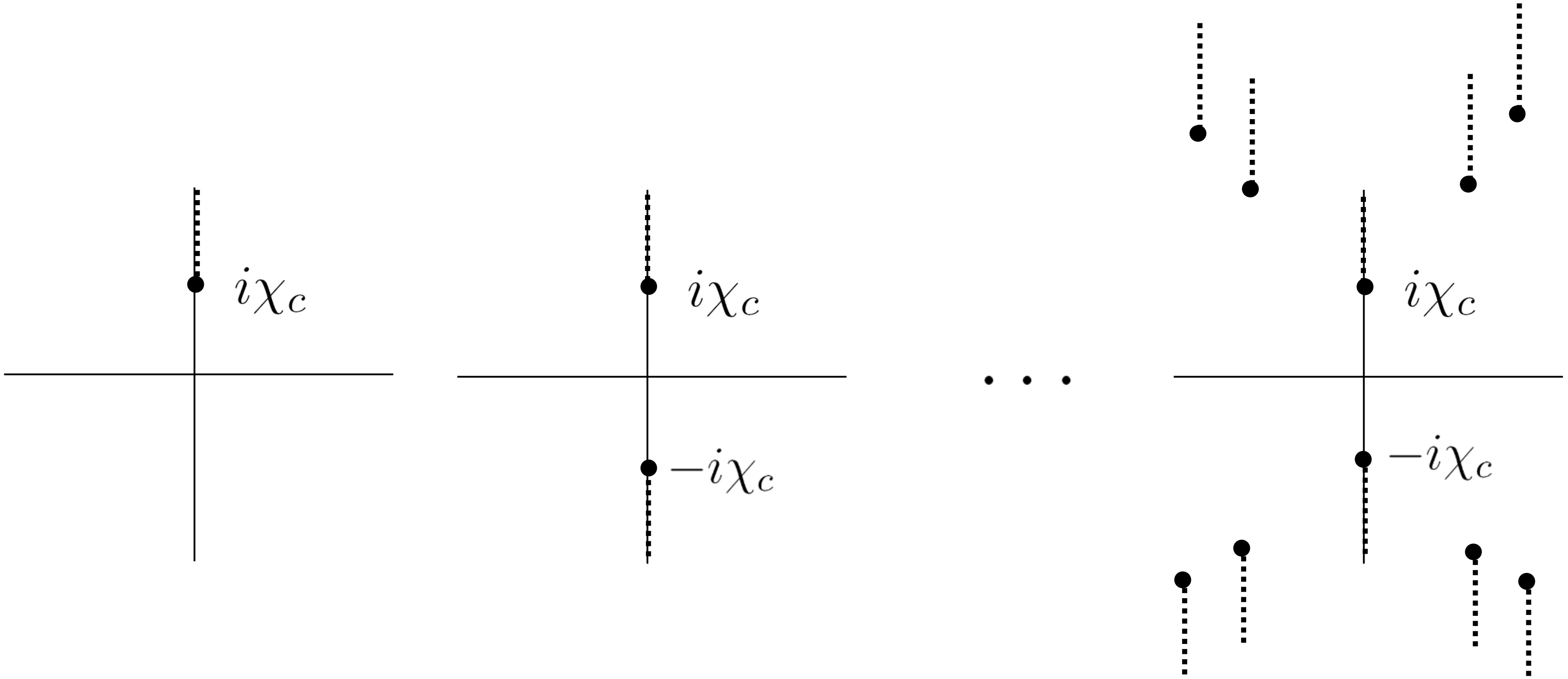
$\alpha_1 \simeq -0.0955383 - \mathrm{i}\,1.8351$ - determined by position of first off-axis singularity

Location of singularities in infinite numbers of sheets of Riemann surface[1]

First (physical) sheet

Second (non-physical) sheet

Third and higher sheets



All singularities are square roots[1]

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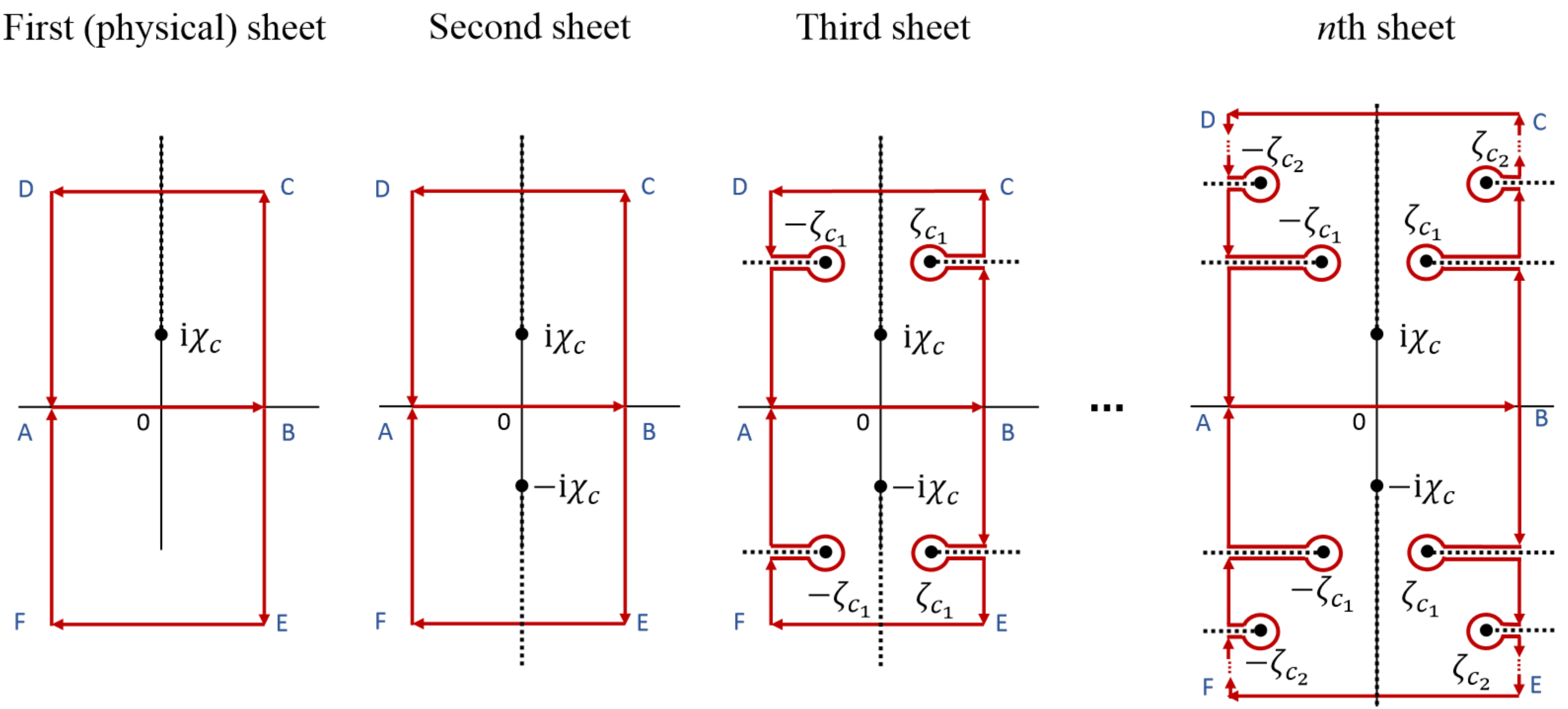
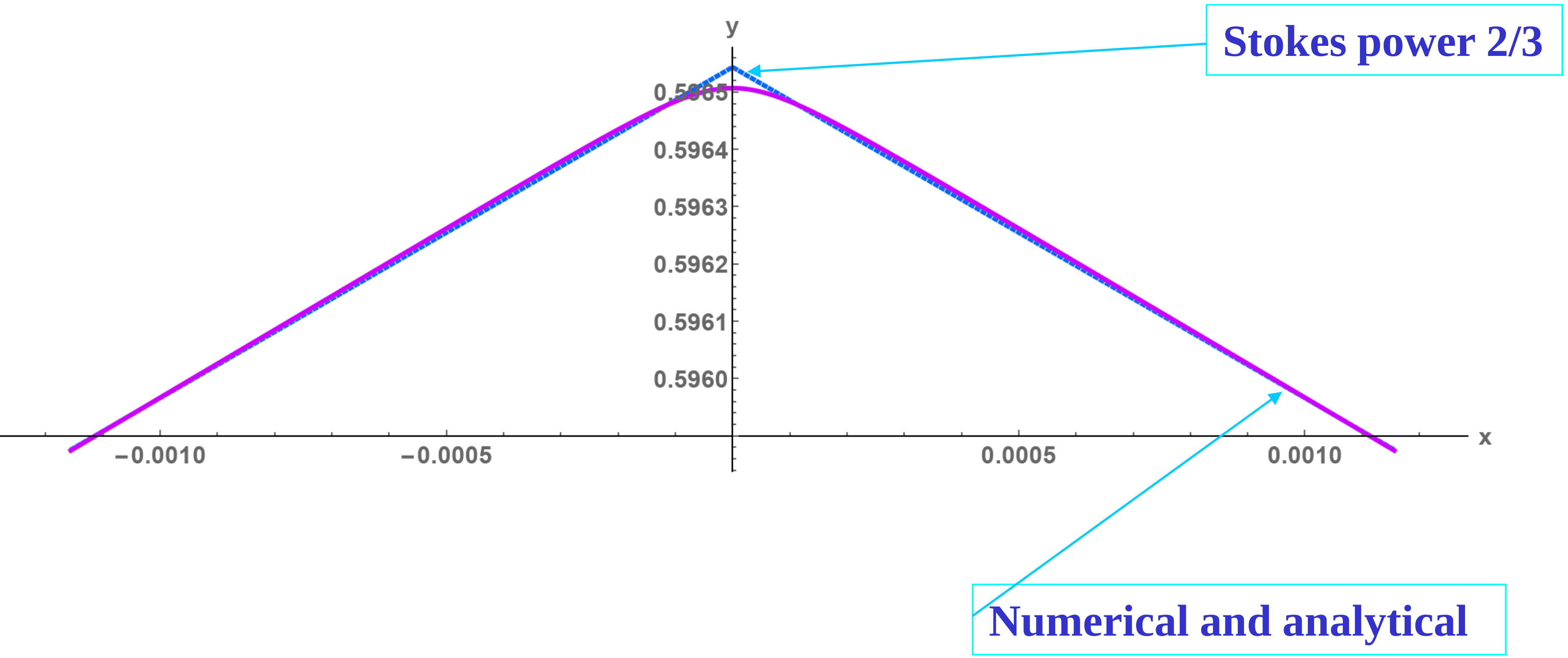
[1]P. M. Lushnikov, Journal of Fluid Mechanics, **800**, 557-594 (2016)

$\alpha_1 \simeq -0.0955383 - \mathrm{i}\,1.8351$ - and all others constants $\alpha_2,\ \alpha_3,\ \alpha_4, \ldots$ are determined by positions of off-axis singularities

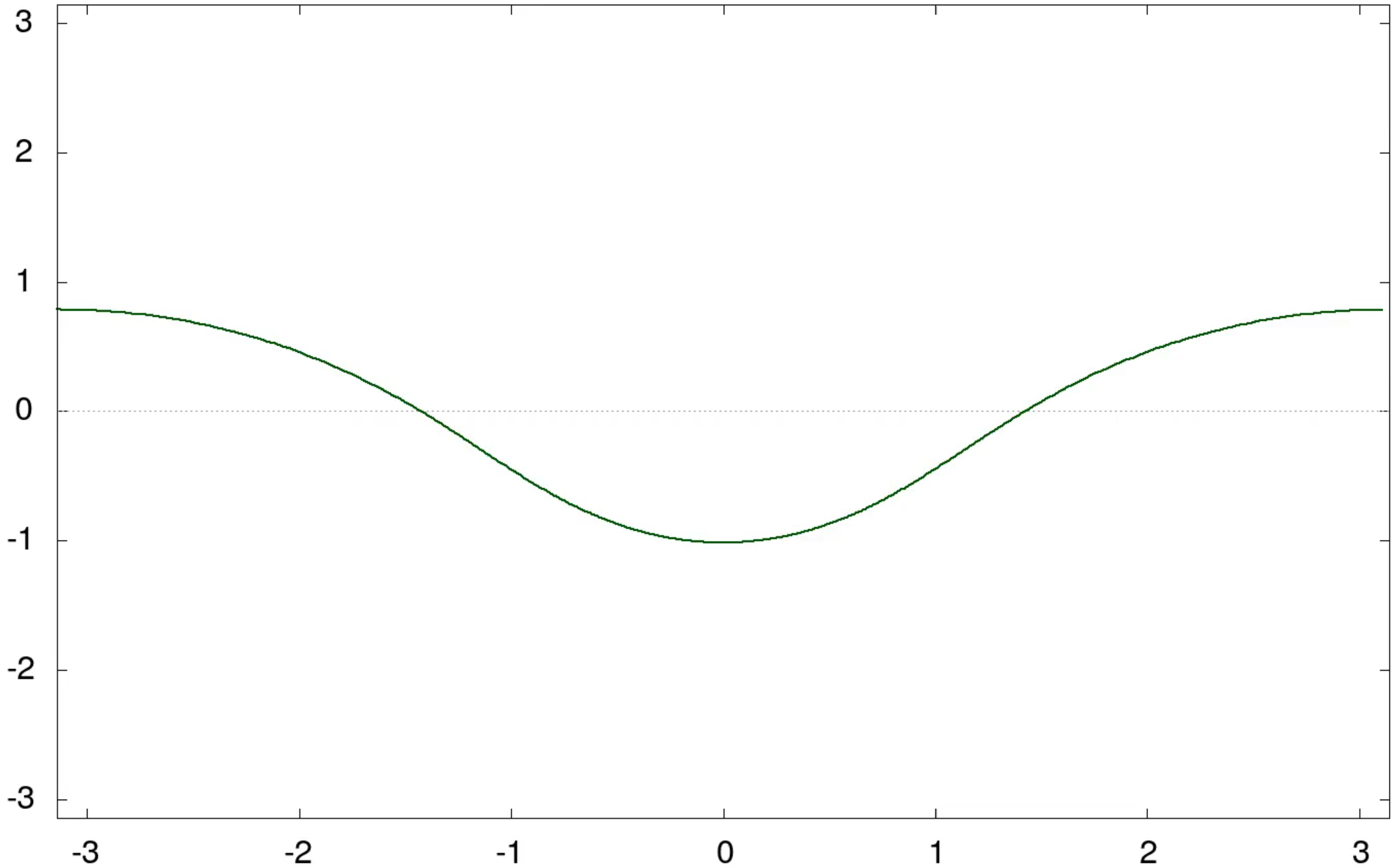


For smaller χ_c analytical and numerical are visually indistinguishable”



$\chi_c \sim 10^{-6}$

Future directions: splash singularity from plunging of overturning wave



Conclusion and future directions

- Non-canonical Hamiltonian equations for the exact free surface dynamics with non-canonical Poisson bracket

- Fully nonlinear quantum Kelvin- Helmholtz instability dynamics is reduced to the Laplace growth equation through the time-dependent conformal map. Laplace growth equation has an infinite number of integrals of motion, the infinite number of exact solutions as well as it is integrable.

- The infinite number of embedded square roots which recovers Stokes limiting wave solution with 2/3 singularity

- For single fluid with free surface we found infinite number of commuting integrals of motion which suggests possible integrability

- Both poles and branch cuts are generic in dynamics

- Number of sheets of Riemann surface is expected to be generally infinite. Construction of pole solutions in different sheets of Riemann surface.

-Ultimate goal for the future is the description of 2D surface motion through the dynamics of coupled poles and branch cuts

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