Expansion of the strongly interacting superfluid Fermi gas: symmetries and self-similar regimes

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OUTLINE

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Motivation: Cooper pairing of fermionic gas and unitarian regime

The discovery of Bose-Einstein condensation (BEC) in alkali gases of bosonic isotopes $^7Li,^{23}Na,^{87}Rb$ in 1995 was awarded by the Nobel prize in 2001. One of the key proofs of the BEC existence was connected with different distribution functions for superfluid and normal components while the gas expansion.
To describe both components it is enough to use the Gross-Pitaevskii (GP) approximation with the Hamiltonian of Bose gas

$$\hat{H} = \int \left[-\hat{\psi}^\dagger \frac{\hbar^2}{2m} \Delta \hat{\psi} + \frac{1}{2} \hat{\psi}^\dagger \hat{\psi}^\dagger g \hat{\psi} \hat{\psi} \right] \mathbf{dr}.$$

where $\hat{\psi}$ and $\hat{\psi}^\dagger$ are bosonic operators and $g = 4\pi\hbar^2 a_s/m$ with a_s being s-scattering length.

Motivation: Cooper pairing of fermionic gas and unitarian regime

Hence the GP equation for the condensate wave function ψ follows after separation:

$$\hat{\psi}(\mathbf{x},t) = \psi(\mathbf{x},t) + \hat{\chi}(\mathbf{x},t),$$

where $\psi(\mathbf{x},t) = \left\langle \hat{\psi}(\mathbf{x},t) \right\rangle$ and the operator $\hat{\chi}(\mathbf{x},t)$, responsible for the non-condensate atoms (normal component), has zero expectation value, $\langle \hat{\chi}(\mathbf{x},t) \rangle = 0$. At $T \to 0$ we have the GPE for ψ.

Depending on the sign of a_s we have different dynamics of the Bose gas. Negative values of a_s leads to attraction of atoms with blow-up behavior while for $a_s > 0$ in optical traps the behavior of the Bose gas is stable.

Motivation: Cooper pairing of fermionic gas and unitarian regime

For Fermi gas (with half-integer spin) $a_s < 0$ provides the formation of Cooper pairs and their condensation of bosonic particles as $T \to 0$. Now there is a big interest to experimental and theoretical studies of such degenerated strongly interacting superfluid Fermi gas in optical traps.

In experiments a_s can be changed by using the Feshbach resonance. In particular, such bosons have very nontrivial behavior when $(|a_s|k_F)^{-1} \to 0$ (k_F is the Fermi momentum) that corresponds to the unitarian regime for which the chemical potential $\mu = (1+\beta)\frac{\hbar^2}{m}\left(6\pi^2 n\right)^{2/3}$ and $\beta = -0.63$. In this case the GPE has the standard form:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2(2m)}\Delta\psi + \mu(n)\psi,$$

where m is a fermion mass ($2m$ is a mass of a fermion pair).

Motivation: Cooper pairing of fermionic gas and unitarian regime

In dimensionless variables this equation coincides with the NLSE, it can be written in the Hamiltonian form

$$i\frac{\partial\psi}{\partial t} = \frac{\delta H}{\delta\psi^*},$$

where Hamiltonian

$$H = \int \left[\frac{1}{2}|\nabla\psi|^2 + |\psi|^{2(\nu+1)}\right] d\mathbf{r},$$

where $\upsilon = 2/3$. Applying the transformation $\psi = \sqrt{n(r,t)}\exp\left(i\varphi(r,t)\right)$ remain the Hamiltonian form for equations for n and φ,

$$\frac{\partial n}{\partial t} = \frac{\delta H}{\delta\varphi}, \quad \frac{\partial\varphi}{\partial t} = -\frac{\delta H}{\delta n},$$

Motivation: Cooper pairing of fermionic gas and unitarian regime

Here the Hamiltonian

$$H = \int \left[\frac{n\left(\nabla\varphi\right)^2}{2} + \frac{\left(\nabla\sqrt{n}\right)^2}{2} + n^{5/3} \right] d\mathbf{r},$$

where the second term is responsible for the quantum pressure. Neglecting this term leads to the quasiclassical equations (Thomas-Fermi approximation) which coincide with the Euler equations for perfect monoatomic gas with $\gamma = 5/3$

$$\frac{\partial n}{\partial t} + (\nabla \cdot n\nabla\varphi) = 0,$$

$$\frac{\partial \varphi}{\partial t} + \frac{(\nabla\varphi)^2}{2} + \frac{5}{3}n^{2/3} = 0,$$

where $\mathbf{v} = \nabla\varphi$ has a meaning of the velocity.

Symmetries and integrals of motion

We would like to remind that the topic of gas expansion was very popular in the hydrodynamic content in 60-s. The first classical works were performed by L.V. Ovsyannikov (1956) and F.J. Dyson (1968).

In 1970 S.I. Anisimov and Yu.I. Lysikov discovered very interesting phenomenon connected with the nonlinear angular deformation of the gas cloud while its expansion. Such behavior directly follows from their remarkable solution for a gas with specific heat ratio $5/3$ based on the symmetry of the dilatation type $\mathbf{r} \to \alpha\mathbf{r}$ and $t \to \alpha^2 t$.

This symmetry (I.E. Dzyaloshinskii, 1970) is well known in quantum mechanics for the potential $V(r) = \beta/r^2$. Indeed, such symmetry first time was exploited by V.P. Ermakov in 1880 to construct solutions for some mechanical systems.

Symmetries and integrals of motion

This $\gamma = 5/3$ is remarkable for both NLSE and its quasiclassical limit. It turns out that the GPE in the unitarian limit have two additional symmetries. The first symmetry forms dilatation group of the scaling type: $\mathbf{r} \to \alpha \mathbf{r}$ and $t \to \alpha^2 t$. For the NLSE such symmetry appears as a result of the conservation of $N = \int |\psi|^2 d\mathbf{r}$ so that at $d = 3$ only the nonlinear potential $\sim |\psi|^{4/3}$ has the same scaling as the Laplace operator Δ. At $d = 2$ such symmetry takes place for the potential $\sim |\psi|^2$ (the stationary self-focusing of light in the Kerr nonlinear media). In the general case, $\nu_{cr} = 2/d$. The second symmetry is of the conformal type first time found by Talanov for the cubic NLSE at $d = 2$.

Symmetries and integrals of motion

These symmetries generate two additional integrals of motion. They can be obtained from the virial theorem

$$\frac{d^2}{dt^2}\int r^2|\psi|^2 d\mathbf{r} = 4H,$$

(first time obtained by Vlasov-Petrishchev-Talanov in 1971 at $d = 2$):

$$\int r^2|\psi|^2 d\mathbf{r} = 2Ht^2 + C_1 t + C_2.$$

Hence we get at $t \to \infty$, independently on C_1 and C_2, linear dependence in time of the r.m.s size of the gas cloud

$$\langle r^2 \rangle^{1/2} \propto t\sqrt{2H/N}.$$

These relations are valid for the GPE and its quasiclassical limit.

Self-similar quasi-classical solution

Let us search for a quasi-classical solution in the self-similar form

$$n = \frac{1}{a_x a_y a_z} f\left(\frac{x}{a_x}, \frac{y}{a_y}, \frac{z}{a_z}\right)$$

which conserves the total number of particles, assuming scaling parameters a_x, a_y, a_z to be functions of t. Then the continuity equation admits integration

$$\varphi = \varphi_0(t) + \sum_l \frac{\dot{a}_l a_l}{2} \xi_l^2.$$

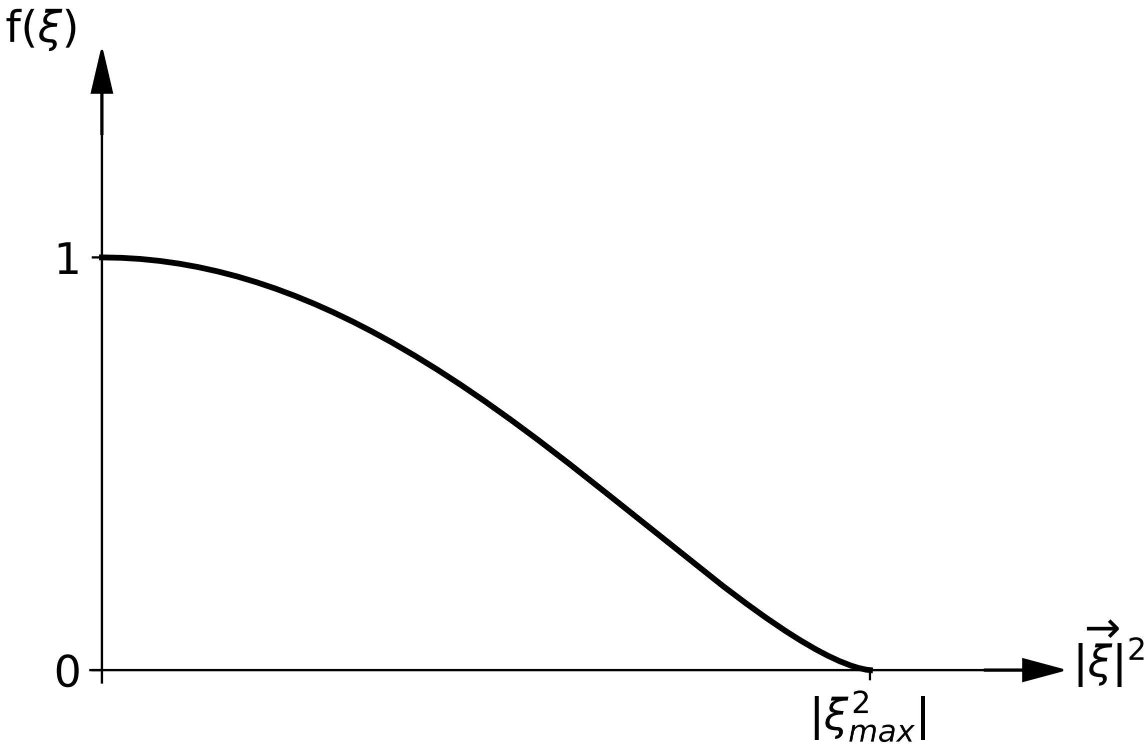
Substitution φ in the eikonal equation yields 3 ODEs which are the Newton equations for motion of a particle

$$\ddot{a}_i = -\frac{\partial U}{\partial a_i}, \;\; U = \frac{3\lambda}{2\left(a_x a_y a_z\right)^{2/3}}.$$

Self-similar quasi-classical solution

Here constant $\lambda > 0$ is found from the initial condition. The density is

$$n = \frac{1}{a_x a_y a_z}\left[1 - \frac{3\lambda}{10}\xi^2\right]^{3/2}.$$



The behavior of the density factor $f(\xi)$ (arbitrary units).

Self-similar quasi-classical solution

The Newton equations have the standard energy integral

$$E = \frac{1}{2} \sum_{i=1,2,3} \dot{a_i}^2 + \frac{3\lambda}{2\left(a_x a_y a_z\right)^{2/3}}.$$

Secondly, for these equations we have the virial identity

$$\frac{d^2}{dt^2} \sum_i a_i^2 = 4E.$$

Its twice integration gives two constants $C_1,\ C_2$. In the spherically symmetric case when $a_x = a_y = a_z \equiv a$, the equations of motion transform into one equation $\ddot{a} = \frac{\lambda}{a^3}$. Its solution shows that gas cloud expands in radial direction at $t \to \infty$ with constant velocity $v_\infty = \sqrt{2E/3}$ (ballistic regime).

Self-similar quasi-classical solution

In the cylindrically symmetric case when $a_x = a_y = a/\sqrt{2}$, $a_z = b$ we have

$$\ddot{a} = -\frac{\partial U}{\partial a}, \ \ddot{b} = -\frac{\partial U}{\partial b}.$$

where $U = 3\lambda 2\left(a^2 b/2\right)^{-2/3}$.

This system belongs to the so called Ermakov type for two degrees of freedom. To integrate this system one needs to have two autonomous integrals of motion in involution. In our case we have three integrals of motion:

$$E = \frac{1}{2}(\dot{a}^2 + \dot{b}^2) + \frac{3\lambda}{2\left(a^2 b/2\right)^{2/3}}.$$

and two constants C_1, C_2. The integrals $C_1, \ C_2$ are not autonomous and can not provide a complete integration.

Self-similar quasi-classical solution

In terms of the polar coordinates $a = r\cos\Phi$, $b = r\sin\Phi$

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\Phi}^2) + \frac{3\lambda}{2^{1/3}r^2\left(\cos^2\Phi\sin\Phi\right)^{2/3}}.$$

The combination $\widetilde{E} = Er^2 - \frac{1}{2}r^2\dot{r}^2 = EC_2 - C_1^2/8$ gives the needed constant (the Ermakov integral) resulting in conservation law for new "energy"

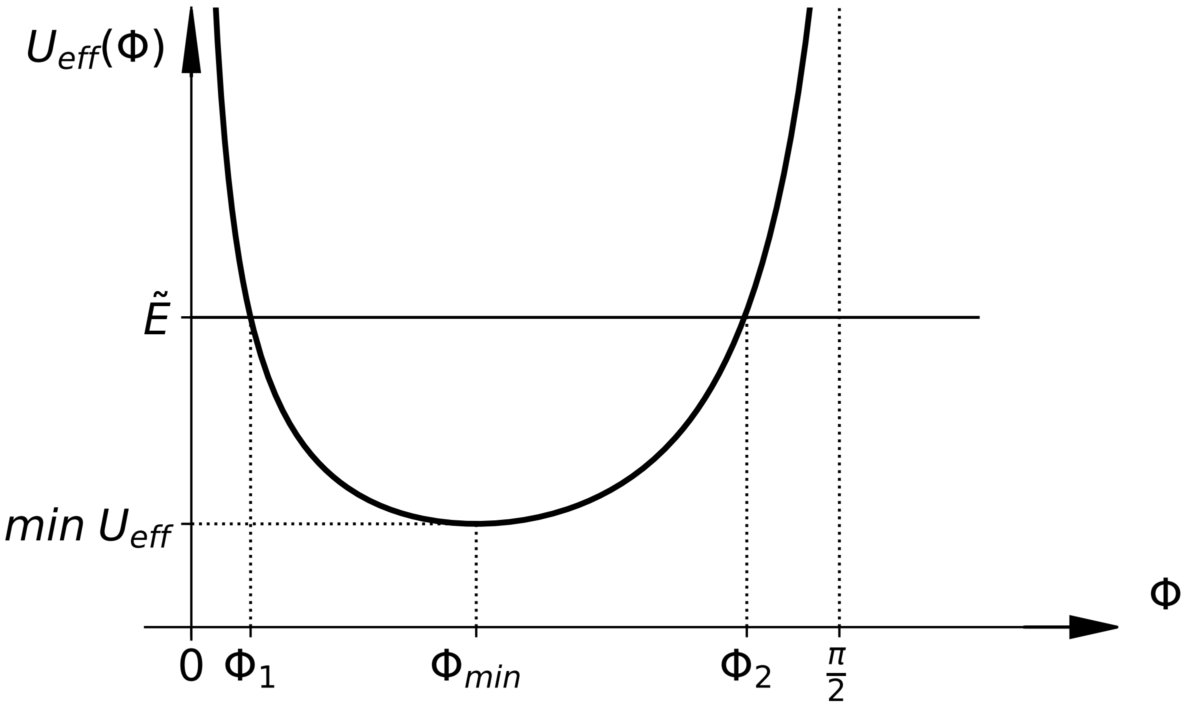
$$\widetilde{E} = \frac{1}{2}\left(\frac{d\Phi}{d\tau}\right)^2 + U_{eff}(\Phi),$$

with new time $\tau = \int_0^t \frac{dt'}{2E(t')^2+C_1t'+C_2}$, where

$$U_{eff}(\Phi) = 3\lambda 2^{1/3}\left(\cos^2\Phi\sin\Phi\right)^{-2/3}.$$

Self-similar quasi-classical solution

Effective potential $U_{eff}(\Phi)$:



The new time τ is expressed through t ,

$$\sqrt{2\widetilde{E}}\tau = \arctan \frac{\sqrt{2E}(t+t_0)}{\chi} - \arctan \frac{\sqrt{2E}t_0}{\chi}$$

where $\chi^2 = \widetilde{E}/E$.

Self-similar quasi-classical solution

If the initial velocity is equal zero $C_1 = 0$ and

$\sqrt{2\widetilde{E}}\tau = \arctan\frac{\sqrt{2\widetilde{E}}t}{C_2}$. In this case, as $t \to \infty$ $\tau \to \tau_\infty = \frac{\pi}{2\sqrt{2\widetilde{E}}}$.

Hence the τ-period of the oscillations in the potential $U_{eff}(\Phi)$ is expressed as

$$T = 2\int_{\Phi^{(-)}}^{\Phi^{(+)}} \frac{d\Phi}{\sqrt{2\left[\widetilde{E} - U_{eff}(\Phi)\right]}},$$

where $\Phi^{(\pm)}$ are reflection points.

At large value of $\widetilde{E}$ oscillations are almost independent on the details of $U_{eff}(\Phi)$. In this case the angular velocity $\frac{d\Phi}{d\tau} \to \pm\sqrt{2\widetilde{E}}$ and the τ -period $T \to \frac{\pi}{\sqrt{2\widetilde{E}}}$.

Self-similar quasi-classical solution

Namely, in this limit T exceeds in two times τ_∞. Notice also that dependence of T with respect to $\widetilde{E}$ is monotonic for the given potential $U_{eff}(\Phi)$. This means that in the real experiment in the better case it is possible one to observe only half of such oscillation, t_{osc}. Thus, a recurrence to the initial shape is impossible in this case. The gas shape behavior will be different for cigar and disk initial conditions. In the cigar case we start from the left reflection point of the potential $U_{eff}(\Phi)$, in the disk case – from the right reflection point. Note that at fixed $\widetilde{E}$ starting from any reflection point we can not reach its opposite reflection point.

The solution presented here was obtained first time by Anisimov and Lysikov in 1970 for expansion of ideal gas with $\gamma = 5/3$.

Self-similar quasi-classical solution

In the general anisotropic case, when all the scaling parameters are different we introduce the spherical coordinates where the Ermakov reduced energy reads

$$\widetilde{E} = C_2 E - \frac{1}{8} C_1^2 = \left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta \left(\frac{d\varphi}{dt}\right)^2 + U_{eff}.$$

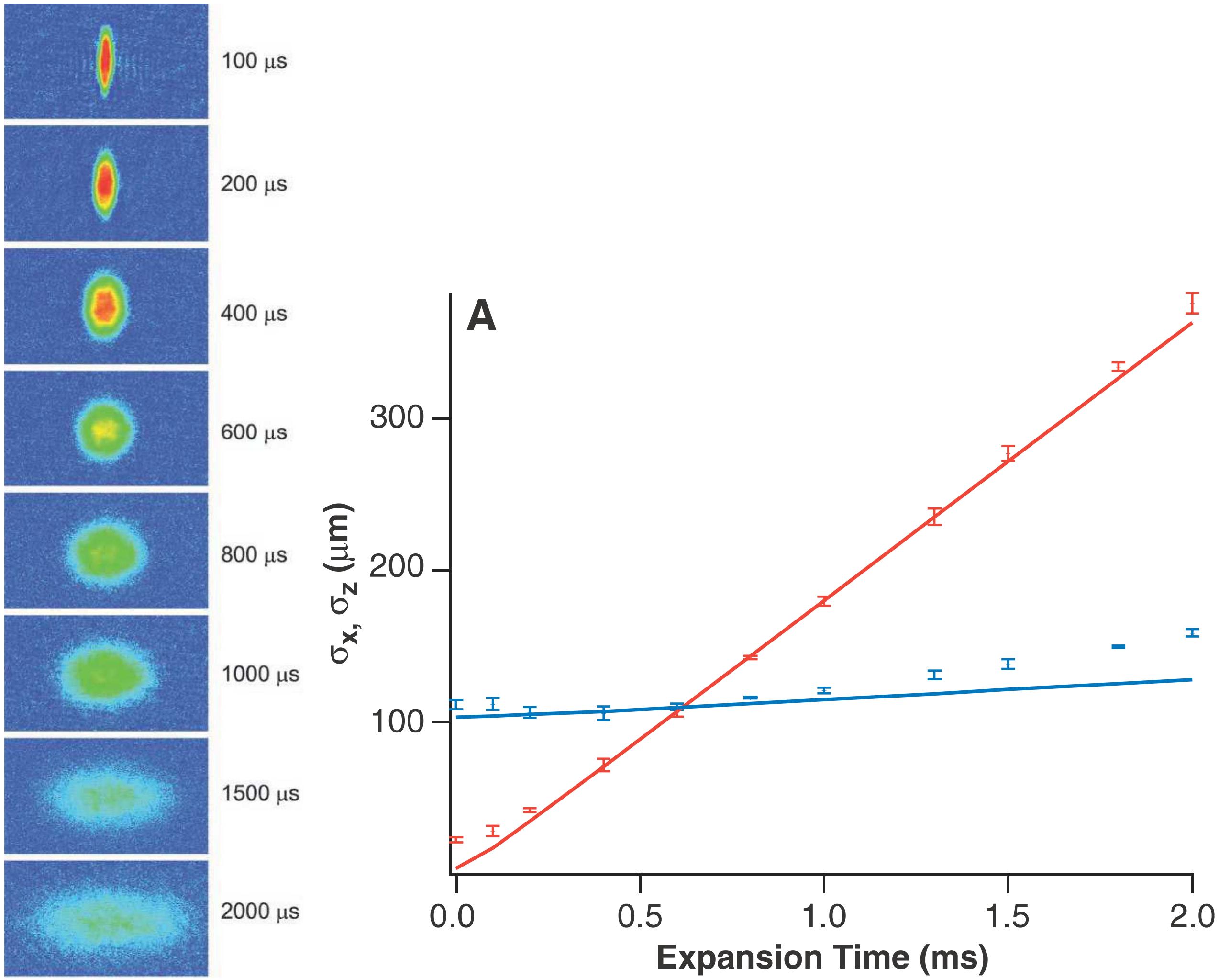
Here

$$U_{eff} = \frac{3\lambda}{2^{1/3}\left(\sin^2\theta\cos\theta\sin 2\varphi\right)^{2/3}}.$$

As it was shown by Gaffet in 1996, this system has one additional integral which follows from the Painleve test. As in the previous limit motion in this potential remains its nonlinear quasi-oscillation character.

Comparison with experimental data

The self-similar expansion of a strongly interacting Fermi gas was observed by the Thomas group (2002).



Ellipsoid (cigar) $\to$ sphere $\to$ ellipsoid $\perp$ to the initial cigar.

Comparison with experimental data

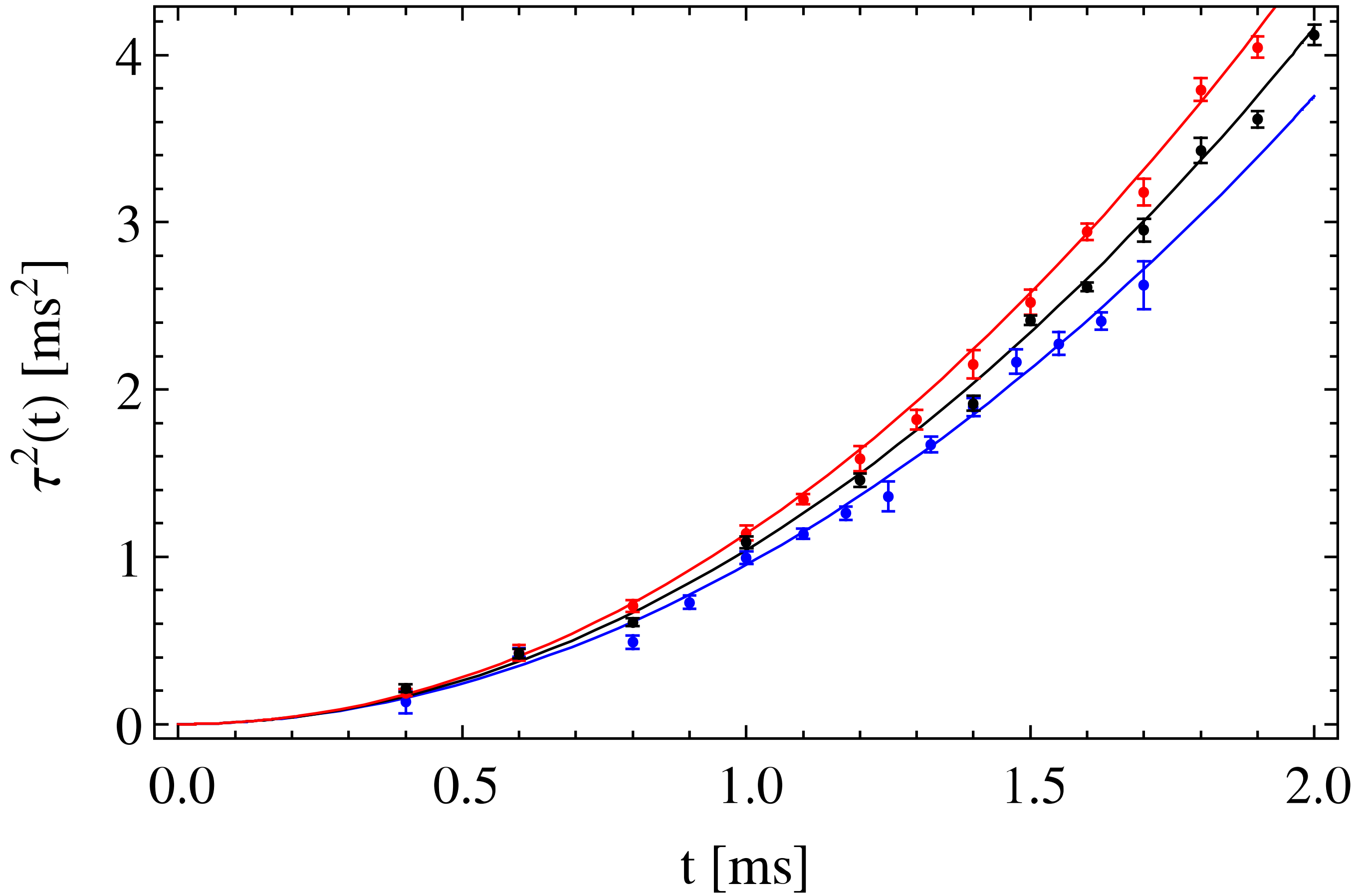
Exactly on resonance, the mean squared cloud size $\langle \mathbf{r}^2 \rangle \equiv \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle$ is found (2014) to evolve as

$$\langle \mathbf{r}^2 \rangle = \langle \mathbf{r}^2 \rangle_{t=0} + \frac{t^2}{m} \langle \mathbf{r} \cdot \nabla U(\mathbf{r}) \rangle_{t=0},$$

where $U(\mathbf{r})$ is the initial trapping potential. This expansion law coincides with the quasi-classical $\langle \mathbf{r}^2 \rangle$ in the unitarian limit. When the system is far from the unitarian point $(k_F a_s)^{-1} = 0$ experiments nevertheless give the parabolic time dependence for $\langle \mathbf{r}^2 \rangle$. Small deviation of the data from the self-similar behavior has been attributed to the contribution of quantum pressure. This difference may be explained since in experiment the interaction parameter is not tuned exactly on resonance $1/(k_F a_s) = 0$, with the estimate $1/(k_F a_s) \simeq -0.14$.

Comparison with experimental data

Experimental values $\tau^2(t) \equiv m[\langle \mathbf{r}^2 \rangle - \langle \mathbf{r}^2 \rangle_{t=0}]/\langle \mathbf{r} \cdot \nabla U \rangle_{t=0}$



Black markers correspond to the gas on resonance, $1/(k_F a_s) = 0$, red and blue markers to $1/(p_F a_s) \simeq 0.59$ and $1/(k_F a_s) \simeq -0.61$.

Conclusion

- We have demonstrated that symmetry for the GPE in the unitarian limit, describing strongly interacting superfluid Fermi gas, provides existence of the virial theorem.
- Independently on the ratio between quantum pressure and chemical potential while the Fermi superfluid gas expansion the size of the gas cloud scales linearly with time asymptotically as $t \to \infty$ with constant velocity $v_\infty = (2H/N)^{1/2}$.

Conclusion

- For description of the Fermi gas expansion in the quasiclassical limit (the Thomas-Fermi approximation) we have constructed the self-similar anisotropic solution. For large time scales the theory matches quite well with simple ballistic ansatz and also with the initial quasi-classical distribution of trapping gas.
- For the initial condition in the cigar-shape form the self-similar solution demonstrates successively all the stages of gas expansion, starting from the distribution extended along the cigar axis, bypassing the spherically symmetrical one and ending with the distribution, turned at angle $\pi/2$ with respect to the initial cigar form. Such behavior was observed first time in experiments.

HAPPY BIRTHDAY TO YOU, ISAAK MARKOVICH!