Computational complexity of full counting statistics of quantum particles in product states

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[1] D.A.Ivanov, arXiv:1603.02724, Phys. Rev. A 96, 012322 (2017).
[2] D.A.Ivanov and L.Gurvits, arXiv:1904.06069.

Main results and motivation

Results:

some results concerning the computational complexity
of expectation values of the form

$$\langle\psi_0| \otimes \ldots \otimes \langle\psi_0| \; e^{a_{ij} c_i^\dagger c_j} \; |\psi_0\rangle \otimes \ldots \otimes |\psi_0\rangle$$

Motivation:

- Quantum computing $\longrightarrow$
- Full counting statistics in quantum systems

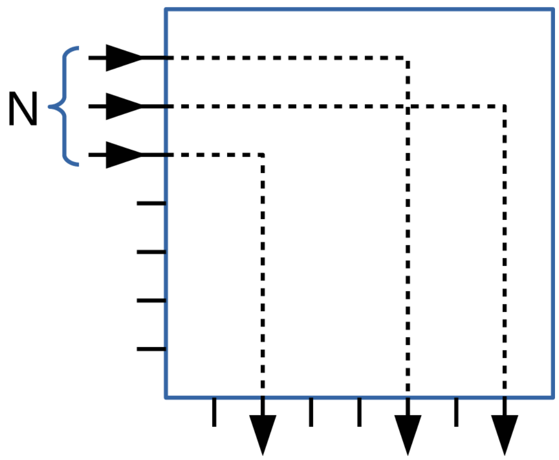
- Theoretical aspects (algorithms, complexity classes)
- Hardware ???

Possibly, a quantum device dedicated to a specific problem
instead of a universal quantum computer?

“Quantum supremacy” proposals

Useless problems that can be solved/simulated on quantum devices, but difficult to model by classical means

Boson sampling [S.Aaronson, 2011]

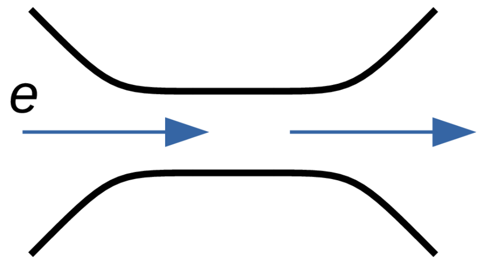


Scattering of non-interacting bosons is hard to model on classical computers

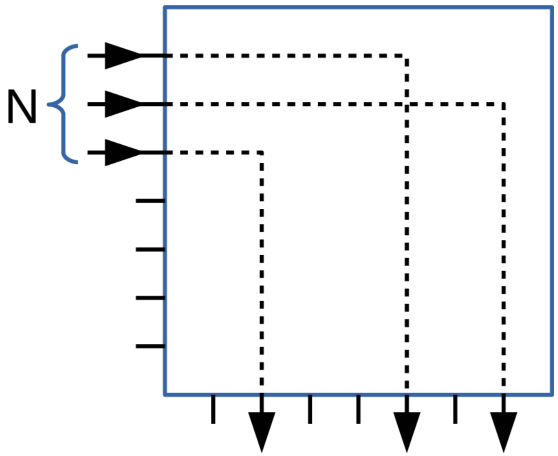
Reason: amplitudes are **permanents**, which are (presumably) hard to calculate [Valiant, 1979] (related to P $\neq$ NP)

Note: sampling is NOT equivalent to calculating amplitudes or probabilities, so a connection between the complexity of amplitudes and the complexity of sampling is more subtle

Full counting statistics



Counting individual electrons passing through a microcontact: effects of fermionic statistics [Levitov, Lesovik 1994]



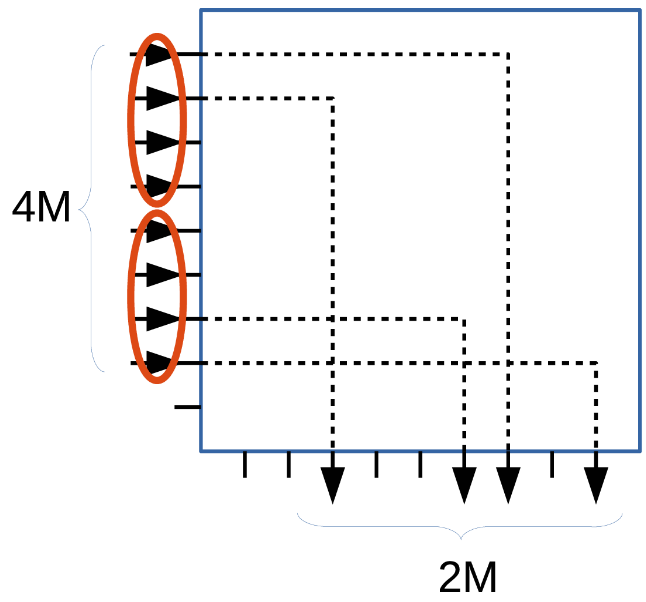
Many studies used free fermion models → amplitudes and generating functions can be expressed as **determinants**, easy to compute ($O(N^3)$ operations)

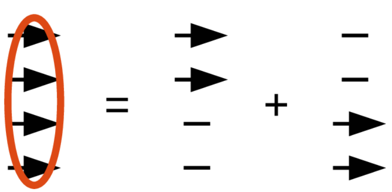
Are bosons more complicated than fermions?
(where are we cheated?)

Resolution of the “paradox”

In Boson Sampling, single-particle states are “quantum”
(non-Gaussian, Wick theorem fails),
which is presumably the source of computational complexity

To test this explanation: try non-Gaussian fermionic states
Simplest non-Gaussian fermionic state: $\psi_4 = |1\rangle\,|2\rangle + |3\rangle\,|4\rangle$





Such amplitudes are at least as hard as permanents (#P hard)
Direct proof: DI 2016
or reduction to an earlier result of L.Gurvits on mixed discriminants

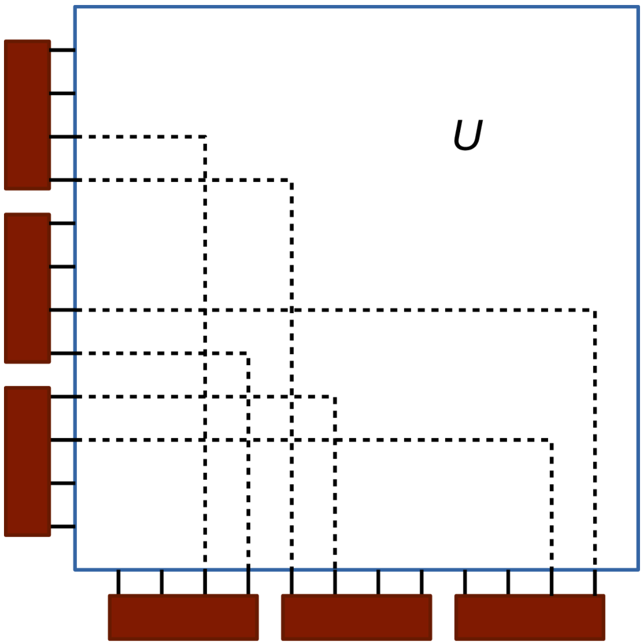
A more general formulation

An “elementary” boson/fermion state ψ_0 may be classified as “**easy**” or “**hard**” depending on the **computational complexity** of the expectation value

$$\langle\Psi|\,\hat{U}\,|\Psi\rangle \;,$$

where

$$\Psi = \psi_0 \otimes \psi_0 \otimes \ldots \otimes \psi_0 \qquad (\,N \text{ times})$$



Here $\hat{U}$ is a general single-particle operator extended multiplicatively to the multi-particle space. Example: evolution operator (but we don't require unitarity here).

This operator is parameterized by its $O(N^2)$ matrix elements

A few known examples

- ψ_0 is a single-boson state: "**hard**" (permanent, Boson Sampling by Aaronson)
- ψ_0 is a coherent bosonic state: "**easy**" (but requires an exponentiation at the end of the calculation)
- $\psi_0 = \psi_4$: "**hard**" (elementary non-Gaussian quadruplet discussed earlier)
- ψ_0 is any Fermi sea $f_1^\dagger \dots f_m^\dagger |0\rangle$: "**easy**" (a Gaussian state, reduces to determinants)

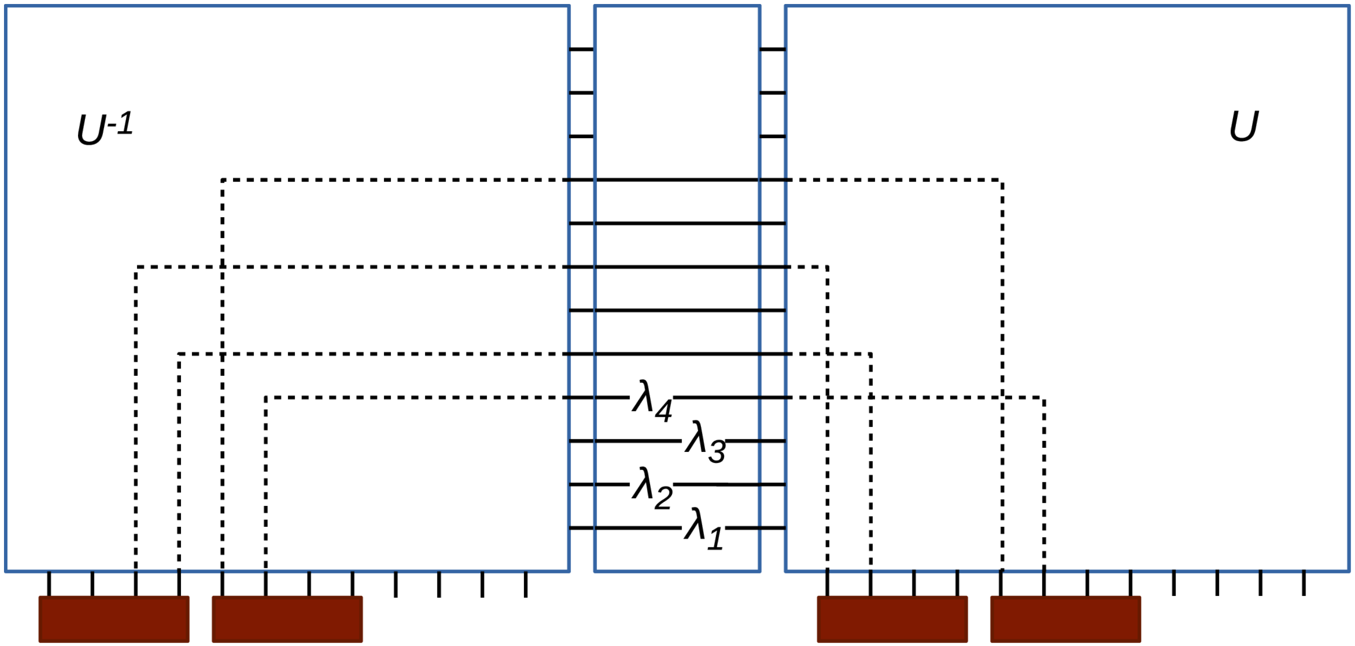
Conjecture: in general, all states ψ_0 are **hard**, except for Gaussian states

Application to full counting statistics

Setup:

1. Prepare the initial multi-particle product state
2. Apply a given non-interacting evolution
3. Measure the generating function of the particle-number probabilities

$$\chi(\lambda_1,\ldots,\lambda_N) = \sum P(n_1,\ldots,n_N)e^{i\lambda_1 n_1+\ldots+i\lambda_N n_N} = \langle\Psi|\,\hat{U}_0^{-1}e^{i\lambda\cdot n}\hat{U}_0\,|\Psi\rangle$$



One may set

$\lambda_i = 0$ to ignore a state and

$\lambda_i \to i\infty$ to impose the zero-particle constraint

Full counting statistics is “hard”

The function $\chi(\lambda_1, \ldots, \lambda_N)$ defined above has the structure $\langle\Psi| \hat{U} |\Psi\rangle$ and therefore is expected to be computationally **hard** for a general non-Gaussian product state Ψ

Disclaimer: a quantum device does not actually allow us to **compute** $\chi(\lambda_1, \ldots, \lambda_N)$ in polynomial time (otherwise we would have built a quantum computer solving NP complete problems, which is very improbable). In fact, interesting values are exponentially small and would still require exponentially many measurements to resolve them.

A variantion of the problem:
What if we only observe a finite number of states?
(keeping all $\lambda_i = 0$, except for a finite number of variables)

Full counting statistics in a small number of states is easy

For a finite number (k) of non-zero variables λ_i ,

$$\hat{U}_0^{-1} e^{i\lambda\cdot n} \hat{U}_0 = \exp\left[\hat{U}_0^{-1} \left(i\lambda \cdot n\right) \hat{U}_0\right] = \widehat{1+V}$$

where V is a **finite-rank matrix** (of rank k)

Then we can prove that computing $\langle\Psi|\,\widehat{1+V}\,|\Psi\rangle$ is **easy**

- for single-boson ψ_0 (= computing $\mathrm{Per}[1+V]$): $O(N^{2k+1})$ operations

- for any fermionic ψ_0 : $O(N^{2k})$ operations

Remains unproven (but probably true): any bosonic ψ_0 with a bounded number of particles

A few technical details of the proofs

Result 1: ψ_4 is "hard"
For any $N \times N$ matrix A, we explicitly construct a $4N^2 \times 4N^2$ matrix U, such that

$$\left\langle (\psi_4)^{N^2} \middle| \hat{U} \middle| (\psi_4)^{N^2} \right\rangle = \mathrm{Per}\, A$$

Result 2: for any fermionic product state, $\langle \Psi | \widehat{1 + \sum_{i=1}^{k} u_i v_i^T} | \Psi \rangle$ is computable in $O(N^{2k})$ operations

The multi-particle operator may be explicitly written in terms of a small number of creation and annihilation operators,

$$\widehat{1 + \sum_{i=1}^{k} u_i v_i^T} = \sum_{\{s_i\}} \hat{u}_{s_1}^\dagger \ldots \hat{u}_{s_r}^\dagger \, \hat{v}_{s_r} \ldots \hat{v}_{s_r}$$

where the sum is over 2^k subsets of indexes, and terms of degree $r \leq k$ are computable in 2^r operations in any fermionic product state

Technical details (continued)

Result 3: $\mathrm{Per}\left(1+\sum_{i=1}^{k} u_i v_i^T\right)$ is computable in $O(N^{2k+1})$ operations

This is done with the help of an auxiliary polynomial of $2k$ variables,

$$F(a_1,\ldots a_k,b_1,\ldots,b_k)=\prod_{x=1}^{N}\left[1+\sum_{s=1}^{k}\sum_{s'=1}^{k} a_s\, b_{s'}\, u_s(x)\, v_{s'}(x)\right]$$
$$=\sum_{\{n_r\},\{n'_r\}} F_{n_1,\ldots,n_k,n'_1,\ldots,n'_k}\, a_1^{n_1}\ldots a_k^{n_k}\, b_1^{n'_1}\ldots b_k^{n'_k}$$

The permanent may be expressed in terms of its diagonal coefficients:

$$\mathrm{Per}\left(1+\sum_{i=1}^{k} u_i v_i^T\right)=\sum_{\{n_r\}} F_{n_1,\ldots,n_k,n_1,\ldots,n_k}\prod_{r=1}^{k} n_r!$$

Summary and comments

- Computational complexity of quantum-mechanical amplitudes may help in identifying setups potentially useful for quantum computing

- Full-counting-statistics (FCS) generating functions may be hard to compute even for non-interacting particles in product states, if those states are non-Gaussian

- FCS generating functions restricted to a small number of states are easy to compute in a few considered examples of product states (single-boson and any fermionic product states)