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Exact solutions in General Relativity, Kasner universes and singularities.

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Content

Motivation

It is due to the influence of Isaak Marcovich Khalatnikov that I became interested in

- Exact solutions of the Einstein equations in General Relativity
- Kasner solution and Bianchi-I geometries
- Singularities in General Relativity and Cosmology

Some history

- 1916–K. Schwarzschild has found static spherically symmetric solutions of the Einstein equations. The singularity arises and disappears.
- 1917–H. Weyl has found a solution of the Einstein equations with plane symmetry. The singularity is present.
- 1918–T. Levi-Civita discovers the same solution.
- 1921–E. Kasner discovers the solution for which the Weyl-Levi-Civita solution is a particular case.
- 1951–A. Taub rediscovers the Kasner solution.
- 1959–O. Heckmann and E. Schucking study the Bianchi-I universe in the presence of dust.
- 1963–I.M. Khalatnikov and E.M. Lifshitz study the Bianchi universes and introduce a convenient parameter for the Kasner solution.

Introduction

- Even in the absence of matter sources the Einstein equations of General relativity can have very nontrivial solutions with singularities.
- The first such solution was the external Schwarzschild solution for a static spherically symmetric geometry.
- The Schwarzschild solution contains a genuine singularity in the centre of the spherical symmetry.
- To avoid it and to describe real spherically symmetric objects like stars, Schwarzschild also invented an internal solution generated by a ball with constant energy density and with isotropic pressure. At the boundary of the ball the pressure disappears and the external and internal solutions are matched. In this case there is no singularity in the center of the ball.

- The spatial Kasner solution

$$ds^2 = (x - x_0)^{2p_1} dt^2 - dx^2 - (x - x_0)^{2p_2} dy^2 - (x - x_0)^{2p_3} dz^2,$$

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1.$$

  has a singularity at $x = x_0$.
- Is it possible to get rid of it, inserting into the empty spacetime something like a thick slab?
- We have found exact solutions of the Einstein equations for a thick slab with a constant energy density, but, in contrast to the Schwarzschild case the singularity does not vanish.

Einstein equations for spacetimes with spatial geometry possessing plane symmetry

The metric with plane symmetry, where the metric coefficients depend on one spatial coordinate x:

$$ds^2 = a^2(x)dt^2 - dx^2 - b^2(x)dy^2 - c^2(x)dz^2.$$

In the empty spacetime we have two general solutions. One of them is the Minkowski metric, where $a = b = c = 1$ and another one is the Kasner solution:

$$a(x) = a_0(x - x_1)^{p_1},\ b(x) = b_0(x - x_1)^{p_2},\ c(x) = c_0(x - x_1)^{p_3},$$

where $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$.

The Kasner solution is more often used in a "cosmological form":

$$ds^2 = dt^2 - a_0^2 t^{2p_1} dx^2 - b_0^2 t^{2p_2} dy^2 - c_0^2 t^{2p_3}.$$

The study of Kasner dynamics has led to the discovery of the oscillatory approach to the cosmological singularity, known also as the Mixmaster universe. The further development has brought the establishment of the connection between the chaotic behaviour of the universe in superstring models and the infinite-dimensional Lie algebras.

A convenient parametrization of the Kasner indices was presented by Khalatnikov and Lifshitz:

$$p_1 = -\frac{u}{1+u+u^2},\ p_2 = \frac{1+u}{1+u+u^2},\ p_3 = \frac{u(1+u)}{1+u+u^2}.$$

The requirement of symmetry in the plane between the y and z directions implies the condition

$$p_2 = p_3.$$

There are two solutions satisfying this condition. The Rindler spacetime

$$p_1 = 1,\ p_2 = p_3 = 0.$$

The Rindler spacetime represents a part of the Minkowski spacetime rewritten in the coordinates connected with an accelerated observer. There is a coordinate singularity (horizon) at $x = x_1$.

Another solution is Weyl-Levi-Civita solution

$$p_1 = -\frac{1}{3}, \; p_2 = p_3 = \frac{2}{3}.$$

In terms of the Khalatnikov-Lifshitz parametrization, the Rindler solution corresponds to $u = 0$, while the Weyl-Levi-Civita solution is given by $u = 1$.

The non-vanishing Christoffel symbols are

$$\Gamma^x_{tt} = a'a, \ \Gamma^x_{yy} = -b'b, \ \Gamma^x_{zz} = -c'c,$$
$$\Gamma^t_{tx} = \frac{a'}{a}, \ \Gamma^y_{yx} = \frac{b'}{b}, \ \Gamma^z_{zx} = \frac{c'}{c}.$$

The components of the Ricci tensor are

$$R_{tt} = a''a + \frac{a'b'a}{b} + \frac{a'c'a}{c},$$
$$R^t_t = \frac{a''}{a} + \frac{a'b'}{ab} + \frac{a'c'}{ac},$$
$$R_{xx} = -\frac{a''}{a} - \frac{b''}{b} - \frac{c''}{c},$$
$$R^x_x = \frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c},$$
$$R_{yy} = -b''b - \frac{a'b'b}{a} - \frac{b'c'b}{c},$$
$$R^y_y = +\frac{b''}{b} + \frac{a'b'}{ab} + \frac{b'c'}{bc},$$

$$R_{zz} = -c''c - \frac{a'c'c}{a} - \frac{b'c'c}{b},$$
$$R^z_z = \frac{c''}{c} + \frac{a'c'}{ac} + \frac{b'c'}{bc}.$$

The Ricci scalar is

$$R = 2\left(\frac{a''}{a} + \frac{b''}{b} + \frac{c''}{c} + \frac{a'b'}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc}\right).$$

The energy-momentum tensor for a fluid with isotropic pressure is

$$T_{\mu\nu} = (\rho + p(x))u_\mu u_\nu - p(x)g_{\mu\nu},$$
$$\rho = \frac{4k^2}{3} = \text{constant}$$
$$u_t = a,\ u_x = u_y = u_z = 0.$$

The equation

$$T^{\nu}_{\mu;\nu} = 0$$

for $\mu = x$ gives

$$p' = -\frac{a'}{a}(\rho + p),$$

$$p = -\frac{4k^2}{3} + \frac{p_0}{a},$$

The Einstein equations are

$$-\frac{b''}{b} - \frac{c''}{c} - \frac{b'c'}{bc} = \frac{4k^2}{3},$$

$$\frac{a'b'}{ab} + \frac{a'c'}{ac} + \frac{b'c'}{bc} = p,$$

$$+\frac{a''}{a} + \frac{c''}{c} + \frac{a'c'}{ac} = p,$$

$$+\frac{a''}{a} + \frac{b''}{b} + \frac{a'b'}{ab} = p.$$

Introducing new functions

$$A = \frac{a'}{a}, \;\; B = \frac{b'}{b}, \;\; C = \frac{c'}{c},$$

we can rewrite the Einstein equations

$$-B' - B^2 - C' - C^2 - BC = \frac{4k^2}{3},$$

$$AB + AC + BC = p,$$

$$A' + A^2 + C' + C^2 + AC = p,$$

$$A' + A^2 + B' + B^2 + AB = p.$$

Solution with isotropic pressure

We consider a slab with $-L \leq x \leq L, -\infty < y, z < \infty$.

$$B = C,$$

$$-2B' - 3B^2 = \frac{4k^2}{3}.$$

Integrating this equation, we obtain

$$B = C = -\frac{2}{3} k \tan k(x + x_0).$$

$$b = b_0 (\cos k(x + x_0))^{\frac{2}{3}},$$

$$c = c_0 (\cos k(x + x_0))^{\frac{2}{3}}.$$

In order to not have singularities inside the slab, we require

$$[-L + x_0, L + x_0] \subset (-\pi/2, \pi/2).$$

For the scale factor a:

$$-\frac{a'}{a}\frac{4k}{3}\tan k(x+x_0) + \frac{4k^2}{9}\tan^2 k(x+x_0) = -\frac{4k^2}{3} + \frac{p_0}{a}.$$

$$a' - \frac{k}{3}\tan k(x+x_0)a - k\cot k(x+x_0)a$$
$$+\frac{3p_0}{4k}\cot k(x+x_0) = 0.$$

The general solution is

$$a(x) = \frac{3p_0}{4k^2}\cos^2 k(x+x_0)$$
$$+\frac{p_0}{2k^2}(\cos k(x+x_0))^{\frac{1}{3}}|\sin k(x+x_0)|\mathcal{B}\left(\sin^2 k(x+x_0); \frac{1}{2}, \frac{7}{6}\right)$$
$$+a_3 \sin k(x+x_0)(\cos k(x+x_0))^{-\frac{1}{3}}.$$

where a_3 is an integration constant and

$$\mathcal{B}(x,r,s) \equiv \int_0^x du u^{r-1}(1-u)^{s-1}.$$

is the incomplete Euler Beta-function.

We have two free parameters x_0 and a_3, which we can fix to provide the disappearance of the pressure on the border of the slab.

$$x_0 = L.$$

It guarantees that

$$p(-L) = 0.$$

Hence,

$$2kL < \frac{\pi}{2}.$$

$$a_3 = \frac{p_0}{4k^2}\left(3\sin 2kL\cos^{1/3}2kL - 2\cos^{2/3}2kL\ \mathcal{B}(\sin^2 2kL; 1/2, 7/6)\right).$$

$$a(x) = \frac{3p_0}{4k^2}\cos^2 k(x+L)$$
$$+\frac{p_0}{2k^2}(\cos k(x+L))^{\frac{1}{3}} \sin k(x+L)\mathcal{B}\left(\sin^2 k(x+L); \frac{1}{2}, \frac{7}{6}\right)$$
$$+\frac{p_0}{4k^2}\left(3\sin 2kL \cos^{1/3} 2kL - 2\cos^{2/3} 2kL\ \mathcal{B}(\sin^2 2kL; 1/2, 7/6)\right)$$
$$\times \sin k(x+L)(\cos k(x+L))^{-\frac{1}{3}}.$$

This is a complete solution of the Einstein equations in the slab, where the energy density is constant and the pressure disappears on the boundary between the slab and an empty space.
The scale factors a, b and c and hence the metric coefficients are not even and the solution is not invariant with respect to the inversion

$$x \to -x.$$

Making the change $x \to -x$ we obtain another solution of our equations.
There is no qualitative difference between these two solutions.
The choice $x_0 = \pm L$ is obligatory in order for the pressure to vanish on both boundaries of the slab and, hence, the asymmetry of these two solutions is an essential feature of the problem.
It arises in spite of the initial symmetry of the Einstein equations and of the position of the slab.

Matching of the solutions in the slab with the vacuum solutions outside the slab

Our solution inside the slab possesses symmetry in the plane (y, z). Thus, we shall try to match it at $x < -L$ and at $x > L$ with one of these three solutions: Minkowski, Rindler or Weyl-Levi-Civita.

The plane $x = -L$:

$$a_{\rm ext}(-L) = a(-L),\ b_{\rm ext}(-L) = b(-L),\ c_{\rm ext}(-L) = c(-L),$$
$$a'_{\rm ext}(-L) = a'(-L),\ b'_{\rm ext}(-L) = b'(-L),\ c'_{\rm ext}(-L) = c'(-L).$$

At $x = -L$ the derivatives of b and c disappear, while the derivative of a at this point is different from zero. Thus, we should choose the Rindler geometry for $x < -L$

$$ds^2 = a_R^2(x - x_R)^2 dt^2 - dx^2 - b_R^2(dy^2 + dz^2).$$

At $x = L$ the derivatives of all three scale factors are non-vanishing. Thus, for $x > L$ we have a Weyl-Levi-Civita solution

$$ds^2 = a_{WLC}^2 (x - x_{WLC})^{-2/3} dt^2 - dx^2 - b_{WLC}^2 (x - x_{WLC})^{4/3} (dy^2 + dz^2).$$

More detail:

$$x_R = -L - \frac{3p_0}{4a_3 k^3}.$$

$$x_{WLC} = L + \frac{1}{k} \cot 2kL.$$

If $2kL < \frac{\pi}{2}$ we can't avoid having a singularity in the space on the right side of the slab, at least not if the energy density ρ of the slab is positive. For an infinitely thin slab, the conclusion that the singularity is unavoidable for was obtained in by S. Fulling et al (2015).

Solution with vanishing tangential pressure

Let us consider a more general energy-momentum tensor

$$T_t^t = \rho,\ T_x^x = -p_x,\ T_y^y = -p_y,\ T_z^z = -p_z.$$

The energy-momentum tensor conservation condition

$$p_x' + A(\rho + p_x) + B(p_x - p_y) + C(p_x - p_z) = 0.$$

In our case $B = C$ and $p_y = p_z$. We shall consider the case, where the tangential pressure $p_y = p_z = 0$.

$$A' + A^2 + B' + B^2 + AB = 0.$$

$$a'' - \frac{2}{3}\tan k(x+x_0)a'$$
$$+\left(\frac{4}{3}k^2\tan^2 k(x+x_0) - \frac{2}{3}\frac{k^2}{\cos^2 k(x+x_0)}\right)a = 0.$$

Looking for the solution of this second order linear differential equation in the form

$$a(x)(\cos k(x+x_0))^\alpha(\sin k(x+x_0))^\beta e^{k\gamma(x+x_0)},$$

we find two sets of the parameters giving the solution):

$$\alpha = -\frac{1}{3},\ \beta = 0,\ \gamma = \frac{1}{\sqrt{3}},$$
$$\alpha = -\frac{1}{3},\ \beta = 0,\ \gamma = -\frac{1}{\sqrt{3}}.$$

The general solution

$$a(x) = (\cos k(x+x_0))^{-1/3}\left(a_4 e^{\frac{1}{\sqrt{3}}k(x+x_0)} + a_5 e^{-\frac{1}{\sqrt{3}}k(x+x_0)}\right).$$

$$A = \frac{a'}{a} = \frac{k}{3}\tan k(x+x_0) + \frac{k}{\sqrt{3}}\frac{a_4 e^{\frac{1}{\sqrt{3}}k(x+x_0)} - a_5 e^{-\frac{1}{\sqrt{3}}k(x+x_0)}}{a_4 e^{\frac{1}{\sqrt{3}}k(x+x_0)} + a_5 e^{-\frac{1}{\sqrt{3}}k(x+x_0)}}.$$

The transversal pressure

$$p = -\frac{4k^2}{3\sqrt{3}}\tan k(x+x_0)\frac{a_4 e^{\frac{1}{\sqrt{3}}k(x+x_0)} - a_5 e^{-\frac{1}{\sqrt{3}}k(x+x_0)}}{a_4 e^{\frac{1}{\sqrt{3}}k(x+x_0)} + a_5 e^{-\frac{1}{\sqrt{3}}k(x+x_0)}}.$$

In order to have the pressure vanishing at $x = -L$, we can again choose $x_0 = L$. Then fixing

$$a_5 = a_4 e^{\frac{4kL}{\sqrt{3}}},$$

we have the pressure vanishing also at $x = L$.

$$p = \frac{4k^2}{3\sqrt{3}} \tan k(x+L) \tanh \frac{k}{\sqrt{3}}(L-x),$$

$$a(x) = a_6 (\cos k(x+L))^{-1/3} \cosh \frac{1}{\sqrt{3}} k(x-L).$$

For $x > L$ this solution should be matched with the Weyl-Levi-Civita solution with the same value of the parameter x_{WLC}.
For $x < -L$ the obtained solution is matched with the Rindler solution with

$$x_R = -L + \frac{\sqrt{3}\coth \frac{2kL}{\sqrt{3}}}{k}.$$

In contrast to the case of the Schwarzschild geometry, we have here a non-singular internal solution with an anisotropic pressure, namely with the pressure whose tangental components are identically equal to zero.

Thick slabs and thin shells

Our solutions are non-singular inside the slab if the condition

$$2kL < \frac{\pi}{2}.$$

is satisfied.
If we introduce the notion of the energy of the unit square of the slab M:

$$M = 2\rho L = \frac{8k^2 L}{3},$$

Then

$$L < \frac{\pi^2}{12M}.$$

If we fix the value of M and begin squeezing the slab, diminishing L, we do not encounter anything similar to the Buchdahl limit 1959 for spherically symmetric configurations. If the relation above is satisfied at some value of L_0, it remains satisfied at all finite values of $L < L_0$.

If we start increasing the thickness of the slab then at the value $L = \frac{\pi^2}{12M}$ a singularity arises inside the slab.

What happens when $L \to 0$?
In paper by Geroch and Traschen, 1987 it was proven that the solutions with distributional sources cannot exist for zero-dimensional (point-like particles) and one-dimensional (strings) objects, but can exist for two-dimensional (shells) objects. The reason lies in the non-linearity of the Einstein equations.
The energy density will tend to the delta function

$$\rho_{L\to 0} \to M\delta(x).$$

It was shown by Fulling et al, 2015 that the tangential pressure for a thin shell should also tend to delta function. Is is not so in both our solutions.
Our solutions are well defined but do not have a thin shell limit.

Concluding remarks

- We have found two static solutions for an infinite slab of finite thickness immersed in the spacetime with plane symmetry.
- Our solutions do not have a well-defined thin-shell limit.
- We required that the energy density on the slab is constant and that the pressure disappears at the boundaries of the slab.
- These conditions are the same used in the Schwarzschild internal solution.
- We considered two particular additional conditions: one of them requires the isotropy of the pressure, just like in the Schwarzschild solution , another requires the disappearance of the tangential pressure in all the slab.

- For both these requirements we have found exact solutions.
- One can imagine the existence of a solution where the transversal and tangential pressures are different functions of the coordinate x, vanishing on the borders of the slab.
- One cannot exclude that for some solutions of this kind a smooth transition to the localised matter configurations is possible.
- It would be interesting to find matter distributions, which imply the existence of solutions of the Einstein equations which are matched in the empty regions of the space with the general spatial Kasner solutions.
- One can prove the existence of such solutions, but it is difficult to find them explicitly.

- Singularities is an essential part of the General Relativity.
- It is not always possible to get rid of singularities.
- The crossing of singularities in General Relativity and especially in Cosmology is an important topic.