Anderson localization on random regular graphs: Toy-model of many body-localization

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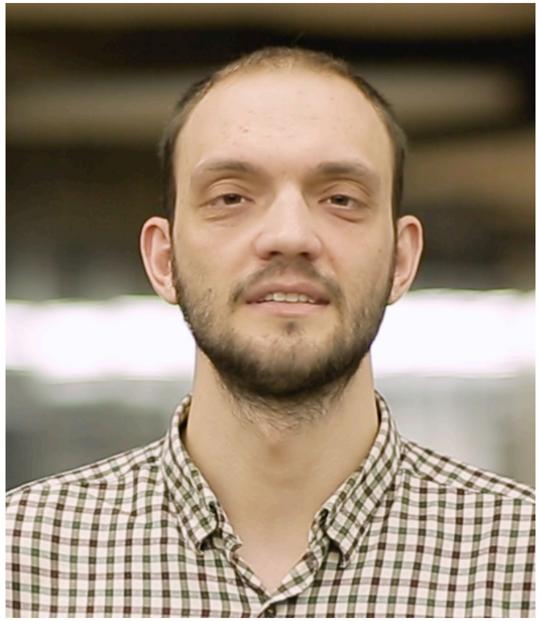
K. S. Tikhonov, ADM, and M.A. Skvortsov, Phys. Rev. B 94, 220203 (2016)
K. S. Tikhonov and ADM, Phys. Rev B 94, 184203 (2016)
M. Sonner, K. S. Tikhonov, and ADM, Phys. Rev. B 96, 214204 (2017)
K. S. Tikhonov and ADM, Phys. Rev. B 97, 214205 (2018)
K. S. Tikhonov and ADM, Phys. Rev. B 99, 024202 (2019)
K. S. Tikhonov and ADM, Phys. Rev. B 99, 214202 (2019)

K. Tikhonov (Moscow, Karlsruhe $\longrightarrow$ Paris)



M. Skvortsov (Moscow)

M. Sonner (Karlsruhe $\longrightarrow$ Geneva)

Anderson localization

Philip W. Anderson

1958 "Absence of diffusion in certain random lattices"

sufficiently strong disorder $\longrightarrow$ quantum localization

$\longrightarrow$ eigenstates exponentially localized, no diffusion

$\longrightarrow$ Anderson insulator

Nobel Prize 1977

Anderson localization

Anderson '58

Quantum particle moving on a lattice:

connectivity K, nearest-neighbor hopping V, disorder W

$$H = \sum_i \epsilon_i c_i^\dagger c_i + \sum_{\langle ij \rangle} V(c_i^\dagger c_j + c_j^\dagger c_i)$$

ϵ_i – random energies, distribution width W

Anderson proved localization for $\quad V < V_c \sim \dfrac{W}{K \ln K}$

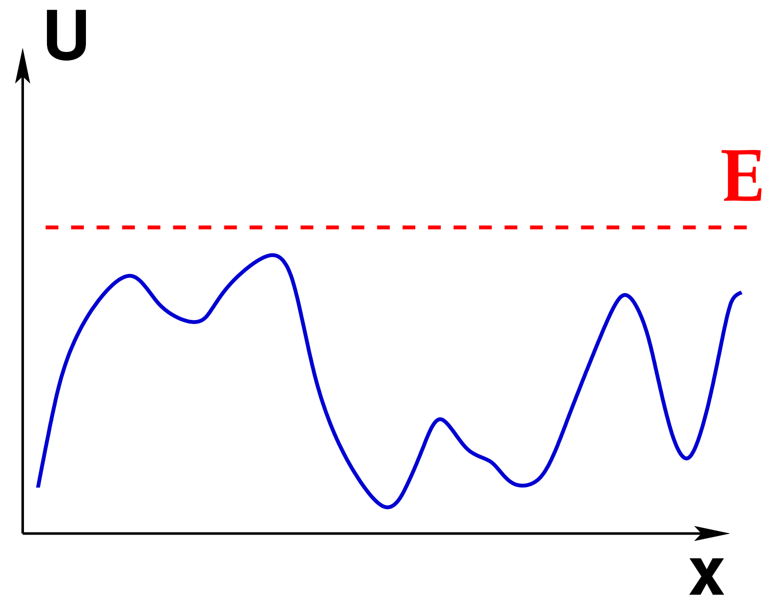
W/K – typical spacing of random energies ϵ_j
of sites directly connected to a given site i

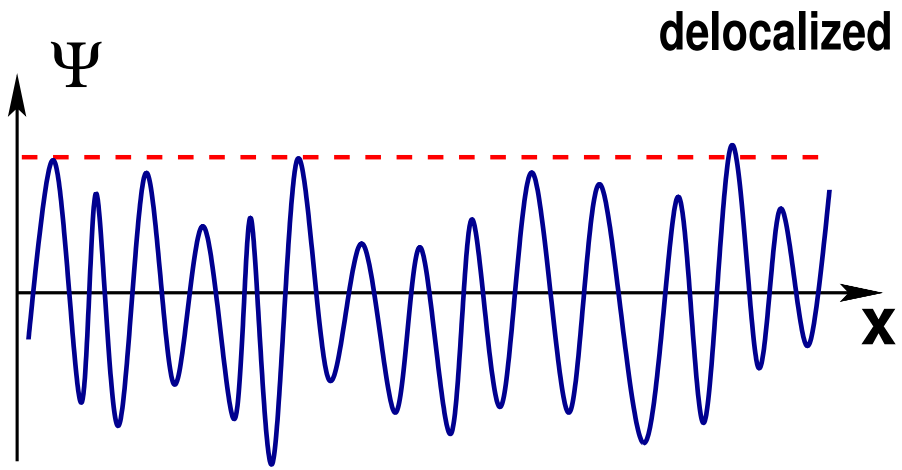
$V \ll W/K \quad \longrightarrow$ hybridization suppressed
$\longrightarrow$ Anderson localization

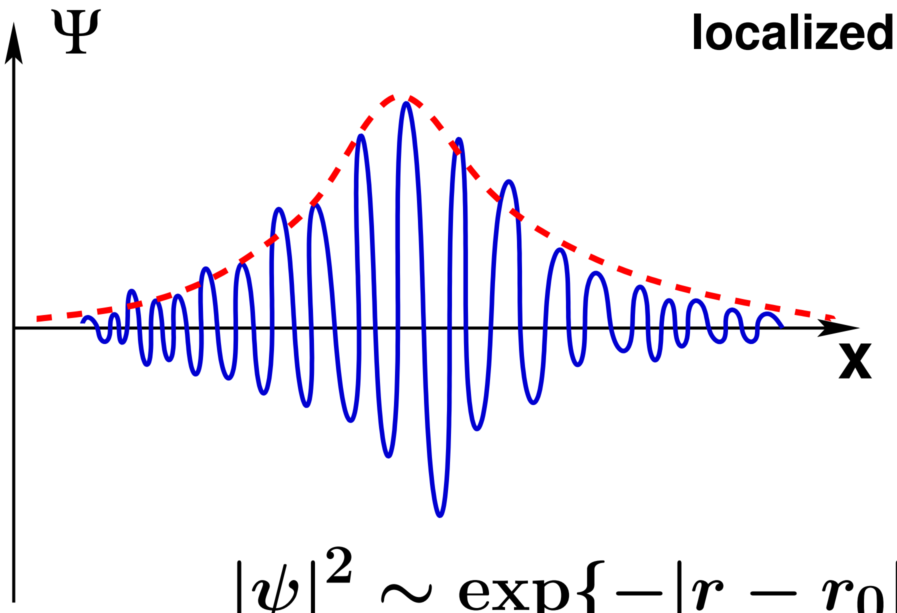
Anderson Localization: Extended and localized wave functions

Schrödinger equation in a random potential

$$[-\hbar^2\frac{\Delta}{2m}+U(\mathbf{r})]\psi = E\psi$$

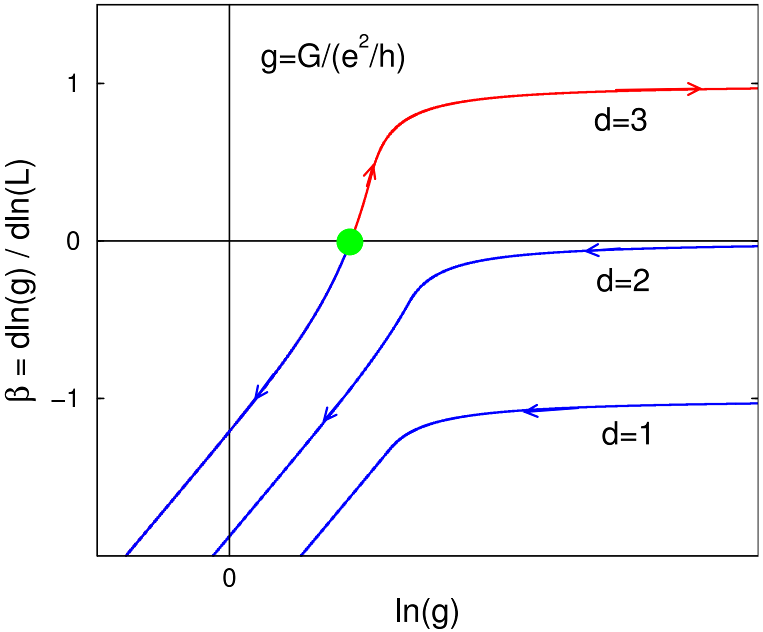






$$|\psi|^2 \sim \exp\{-|r-r_0|/\xi\}$$

Anderson Insulators & Metals

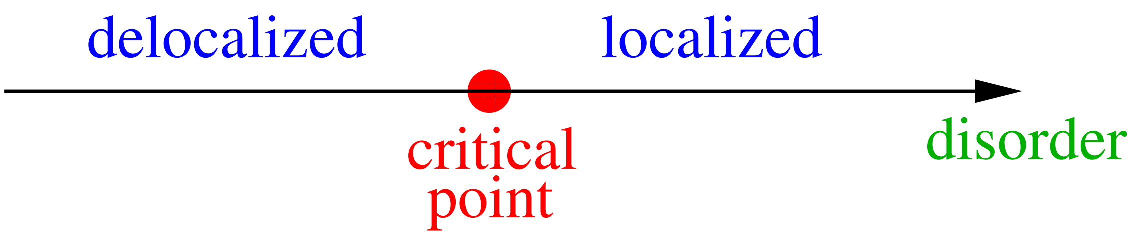


Scaling theory of localization:
Abrahams, Anderson, Licciardello, Ramakrishnan '79

Modern approach:
RG for field theory (σ-model)

quasi-1D, 2D : all states are localized

$d > 2$: Anderson metal-insulator transition



review: Evers, ADM, Rev. Mod. Phys. 80, 1355 (2008)

Many-body localization

Assume that all single-particle states are localized

- **External bath with continuous spectrum (e.g., phonons)**

$\longrightarrow$ inelastic processes $\longrightarrow$ dephasing of quantum interference
$\longrightarrow$ cutoff for localization $\longrightarrow$ thermalization, transport

- **Problem of "many-body localization":**

What happens at finite T in the absence of external bath?
Can the system serve as its own thermal bath?

Early work: Fleishman, Anderson '80:
Inelastic processes inhibited due to discreteness of spectrum;
Localization in many-body space

Many-body localization transition at intermediate T
(or at intermediate disorder at fixed T) for short-range interaction
Gornyi, ADM, Polyakov '05; Basko, Aleiner, Altshuler '06, ...

MBL implies breakdown of ergodicity

Ergodicity and MBL in excited states of many-body systems

Spatially extended systems with short-range interaction

Gornyi, Mirlin, Polyakov, PRL 95, 206603 (2005)
Basko, Aleiner, Altshuler, Ann Phys 321, 1126 (2006)
Oganesyan, Huse, PRB 75, 155111 (2007)

Quantum dots

Altshuler, Gefen, Kamenev, Levitov, PRL 78, 2803 (1997)
Mirlin, Fyodorov, PRB 56, 13393 (1997)
Jacquod, Shepelyansky, PRL 79, 1837 (1997)

Spatially extended systems with power-law interaction

Burin, arXiv:cond-mat/0611387; PRB 91, 094202 (2015)
Yao, Laumann, Gopalakrishnan, Knap, Müller, Demler, Lukin, PRL 113, 243002 (2014)
Gutman, Protopopov, Burin, Gornyi, Santos, Mirlin, PRB 93, 245427 (2016)

and many further papers

$\longrightarrow$ Revival of interest to localization on tree-like graphs

Properties of MBL transition, loc. and deloc. phases, critical regime – ?

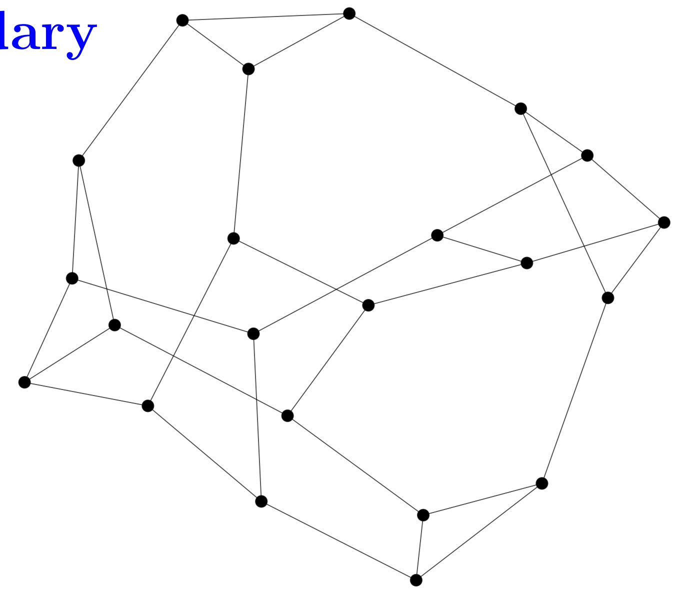
One of important questions: Is the delocalized phase ergodic ?

Anderson localization on random regular graphs (RRG)

Random regular graph – random graph with constant connectivity

Locally tree-like (as Bethe lattice) but without boundary

Typical size of loops $\sim \ln N$



$$\mathcal{H} = \sum_{\langle i,j \rangle} \left(c_i^+ c_j + c_j^+ c_i \right) + \sum_{i=1} \varepsilon_i c_i^+ c_i$$

$\varepsilon_i \longrightarrow$ disorder W

Relation to the MBL problem:

Hilbert space size $N \sim m^L$ where L is "linear size"

Sites $\longleftrightarrow$ many-body basis states, links $\longleftrightarrow$ interaction matrix elements

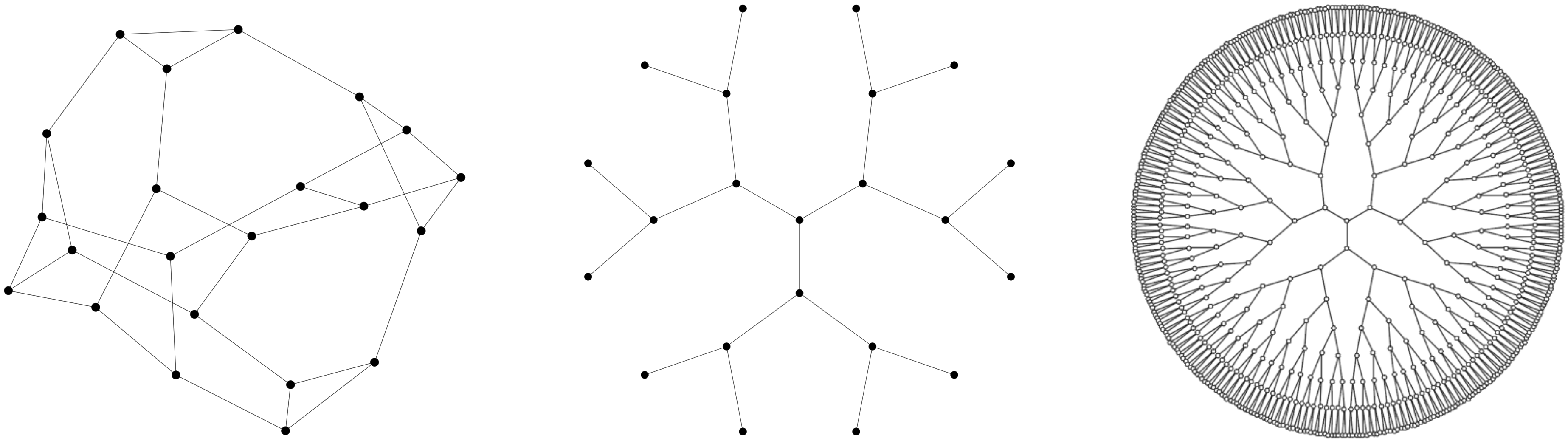
ADM, Fyodorov '91 Supersymmetry theory of Anderson transition in sparse random matrix model ($\sim$ RRG with fluctuating connectivity)

Delocalized phase ($W < W_c$): "ergodicity":

- Wigner-Dyson level statistics
- Wave function statistics: Inverse participation ratio (IPR) $P_2 = \langle \sum_i |\psi(i)^4| \rangle$

$$P_2 \simeq N_c(W)/N \ , \quad \ln N_c \propto (W_c - W)^{-1/2} \ , \quad N \gg N_c$$

RRG vs finite Bethe lattice vs infinite Bethe lattice



RRG: finite N, one can study properties of individuals eigenstates,
e.g. IPR $P_2 = \langle \sum_i |\psi_n(i)|^4 \rangle \longrightarrow$ this talk

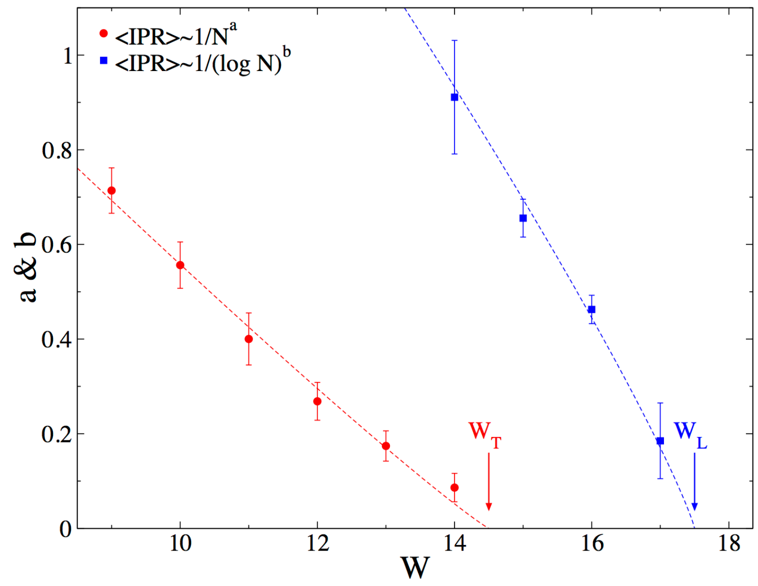
finite BL: finite N, one can study properties of individuals eigenstates,
but they differ crucially from RRG !
Multifractality that depends on W and on position on the tree
Tikhonov and ADM, Phys Rev B 94, 184203 (2016);
Sonner, Tikhonov, and ADM, Phys. Rev. B 96, 214204 (2017)
not considered in this talk

infinite BL: $N = \infty$, one can study statistics of Green functions (e.g. LDOS)
at finite frequency (imaginary or real)

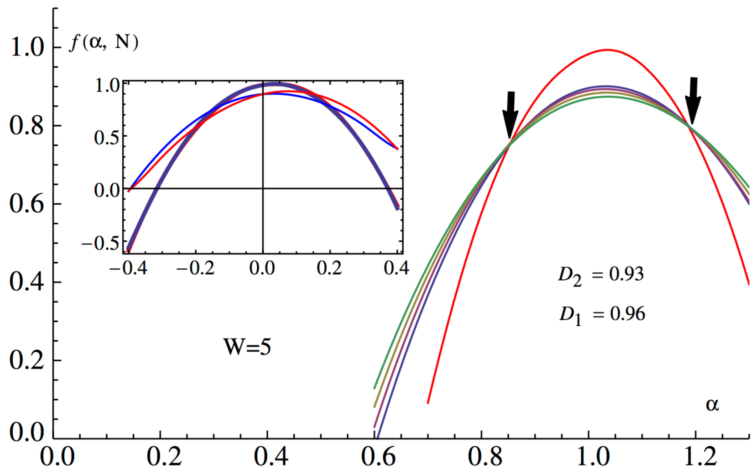
Anderson localization on RRG: Previous numerics

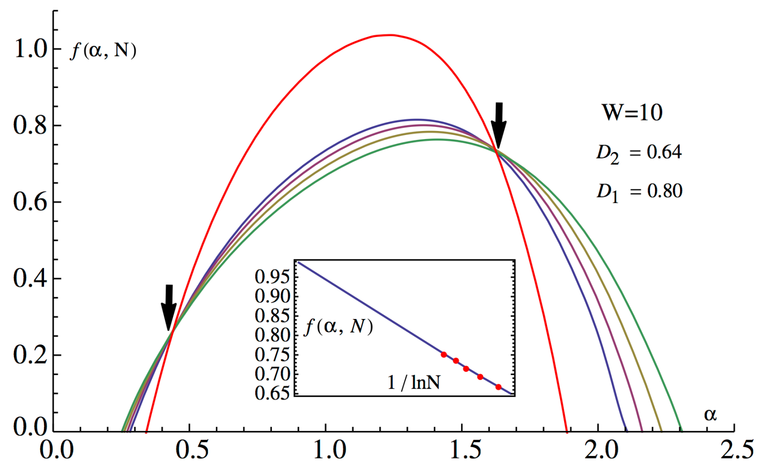
Biroli, Ribeiro-Teixeira, Tarzia,
arXiv:1211.7334

apparent fractality of IPR
⟶ non-ergodictiy of delocalized phase ?!



De Luca, Altshuler, Kravtsov, Scardicchio, Phys Rev Lett '14





"We conclude that the nonergodicity and multifractality persist in the entire region of delocalized states $0 < W < W_c$"

Approaches to Anderson model on RRG

- **Direct numerics: Exact diagonalization**
- **Field theory, Large N** $\longrightarrow$ **saddle point**

  $\longrightarrow$ **self-consistency equation**

  - **Analytical solution**
  - **Numerical solution via pool method (population dynamics)**

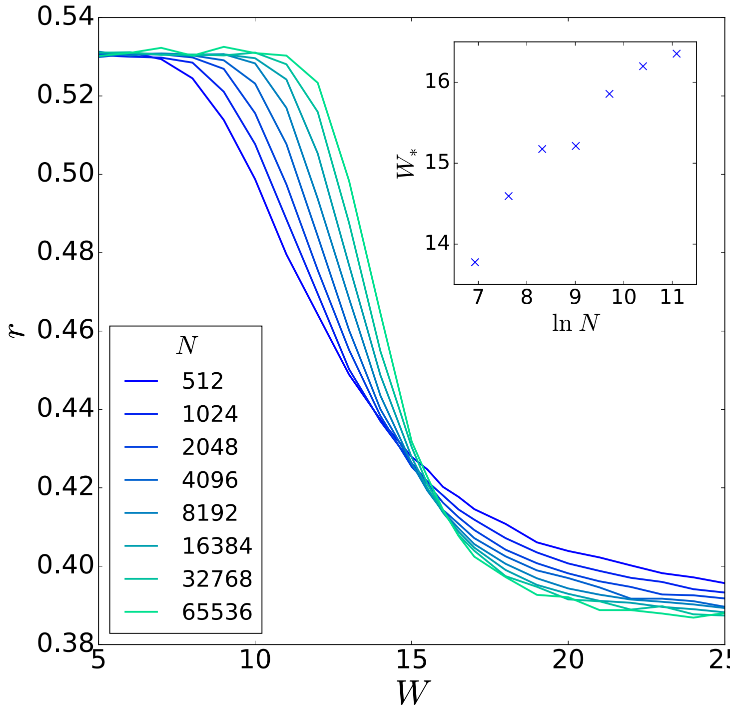
Anderson localization and ergodicity on RRG

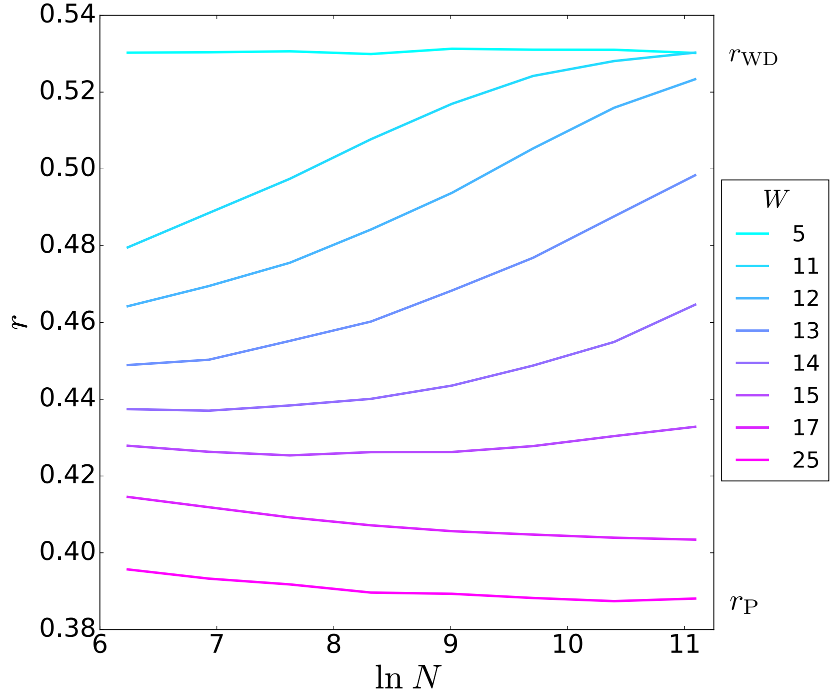
K.S. Tikhonov, ADM, M.A. Skvortsov, PRB 94, 220203(R) (2016)

maximal size $N = 65\,536$; for $W = 11$: $N = 262\,144$

Level statistics:

mean adjacent gap ratio r





Crossing point W_* drifts towards stronger disorder:

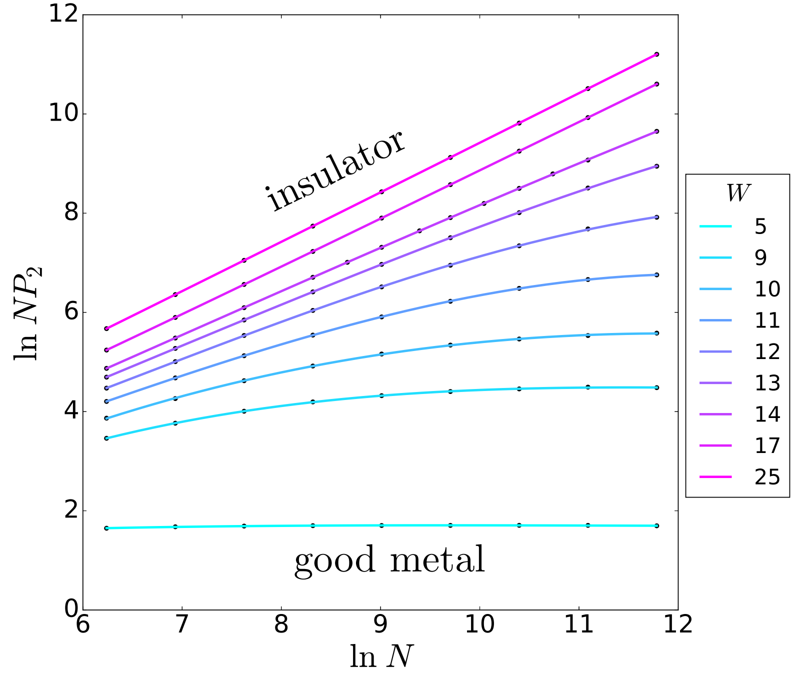
$W_* \simeq 14 \ (N = 512) \longrightarrow W_* \simeq 16 \ (N = 65\,536)$

Equivalently: for given W **non-monotonic dependence** $r(N)$

Explanation: critical point on tree-like structures (or at $d \to \infty$) **has quasi-localized character** (Poisson statistics, IPR $\propto N^0$)

Eigenfunction statistics

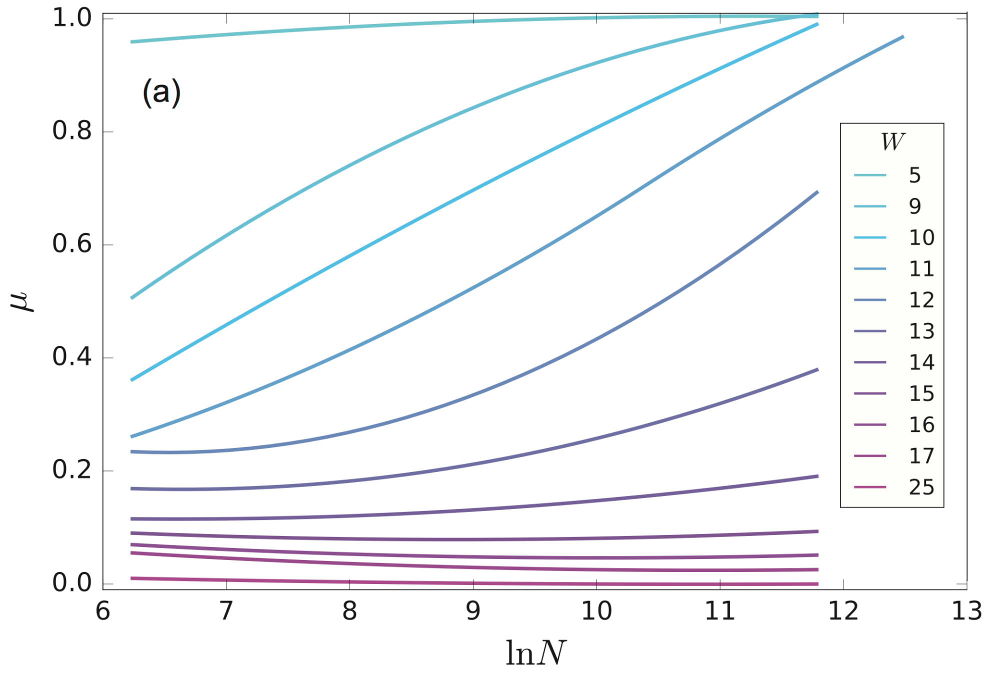
IPR $P_2(W, N)$

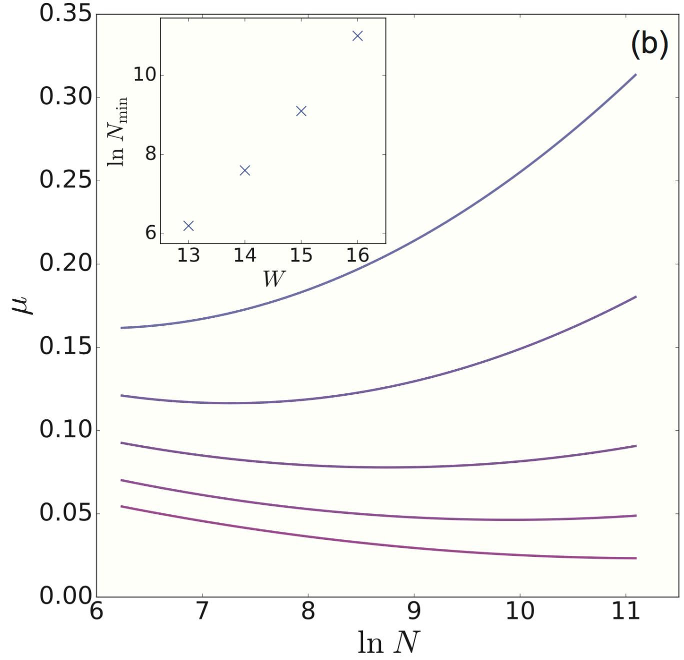


"flowing fractal exponent"

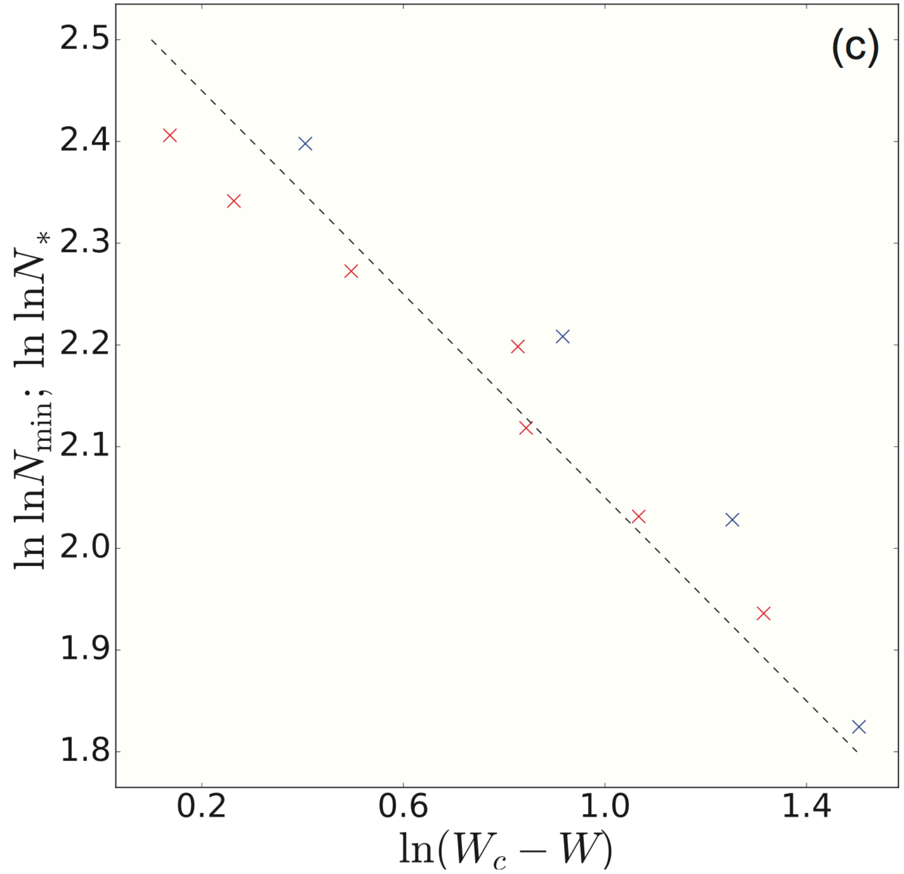
$$\mu(W,N) = -\partial \ln P_2(W,N)/\partial \ln N:$$

non-monotonic N-dependence





Correlation length



× level statistics

× eigenfunction statistics

$$\xi(W) \propto (W_c - W)^{-\nu_d}$$

correlation length

$$N_c(W) \sim m^{\xi(W)}$$

correlation volume

data consistent with $\nu_d = 1/2$

as expected from the critical behavior of IPR (analytics)

$$P_2 \simeq N_c(W)/N \ , \qquad \ln N_c \propto (W_c - W)^{-1/2} \ , \qquad N \gg N_c$$

RRG: Field-theoretical approach

$\langle \mathcal{O} \rangle = \int \prod_k [d\Phi_k] e^{-\mathcal{L}(\Phi)} U_{\mathcal{O}}(\Phi)$ $\quad \Phi_{i,s} = (S^{(1)}_{i,s}, S^{(2)}_{i,s}, \chi_{i,s}, \chi^*_{i,s})$ – supervector

Doubling $\Phi_i = (\Phi_{i,1},\ \Phi_{i,2})$ for retarded (R) and advanced (A) Green functions

$$e^{-\mathcal{L}(\Phi)} = \int \prod_i d\epsilon_i \gamma(\epsilon_i) e^{\frac{i}{2}\Phi_i^\dagger \hat{\Lambda}(E-\epsilon_i)\Phi_i + \frac{i\omega}{4}\Phi_i^\dagger \Phi_i} \prod_{\langle i,j \rangle} e^{-i\Phi_i^\dagger \Phi_j} \qquad \Lambda = \mathrm{diag}(1,-1)_{RA}$$

RRG, connectivity $p = m+1$, distributions of energies $\gamma(\epsilon)$ and hoppings $h(t)$

$$\langle Z \rangle = \int \prod_i d\Phi_i \frac{dx_i}{2\pi} e^{ipx_i} \exp\left\{ \sum_i \left[\frac{i}{2}\Phi_i^\dagger \hat{\Lambda}(E - J_i \hat{K})\Phi_i + \frac{i}{2}\left(\frac{\omega}{2} + i\eta\right) \Phi_i^\dagger \Phi_i \right.\right.$$
$$\left.\left. + \ln \tilde{\gamma}(\frac{1}{2}\Phi_i^\dagger \hat{\Lambda} \Phi_i) \right] + \frac{p}{2N} \sum_{i \neq j} \left[e^{-i(x_i+x_j)} \tilde{h}(\Phi_i^\dagger \hat{\Lambda} \Phi_j) - 1 \right] \right\}$$

Functional generalization of Hubbard-Stratonovich transformation

$\longrightarrow$ integral over functions $g(\Phi)$: $\quad \langle \mathcal{O} \rangle = \int Dg\, U_{\mathcal{O}}(g) e^{-N\mathcal{L}(g)}$

$$\mathcal{L}(g) = \frac{m+1}{2} \int d\Psi d\Psi' g(\Psi) C(\Psi, \Psi') g(\Psi') - \ln \int d\Psi\, F_g^{(m+1)}(\Psi)$$

$$F_g^{(s)}(\Psi) = \exp\left\{ \frac{i}{2} E \Psi^\dagger \hat{\Lambda} \Psi + \frac{i}{2}\left(\frac{\omega}{2} + i\eta\right) \Psi^\dagger \Psi \right\} \tilde{\gamma}(\tfrac{1}{2}\Psi^\dagger \hat{\Lambda} \Psi) g^s(\Psi)$$

Field theory for RRG model: Saddle-point treatment

$\langle \mathcal{O} \rangle = \int Dg\, U_{\mathcal{O}}(g) e^{-N\mathcal{L}(g)}$ Large N $\longrightarrow$ saddle-point treatment

IPR $P_2 = \frac{1}{\pi\nu} \lim_{\eta\to 0} \eta \, \langle G_R(j,j) G_A(j,j) \rangle$ $G_{R,A}(j,j) = \langle j | (E - \mathcal{H} \pm i\eta)^{-1} | j \rangle$

$\langle G_R(j,j) G_A(j,j) \rangle = \int Dg\, U(g) e^{-N\mathcal{L}(g)}$

$U(g) = \int [d\Psi] \, \frac{1}{16} \left(\Psi_1^\dagger \hat{K} \Psi_1 \right) \left(\Psi_2^\dagger \hat{K} \Psi_2 \right) F_g^{(m+1)}(\Psi)$

$g_0(\Psi) = \int d\Phi \, \tilde{h}(\Phi^\dagger \hat{\Lambda} \Psi) F_{g_0}^{(m)}(\Phi)$ saddle-point equation

identical to the self-consistency equation for infinite Bethe lattice (BL) !

ADM, Fyodorov 1991

Symmetry $\longrightarrow$ $g_0(\Psi) = g_0(x,y)$; $x = \Psi^\dagger \Psi$, $y = \Psi^\dagger \hat{\Lambda} \Psi$

Laplace (x) - Fourier (y) transf.: $g_0(x,y) \longleftrightarrow$ distribution of Im G and Re G

self-consistency equation in the form of Abou-Chacra, Thouless, Anderson 1973

Field theory for RRG model: Inverse Participation Ratio

- $W \geq W_c$ localized phase and critical point:

single saddle-point $g_0(\Phi) = g_0(x,y)$, characteristic $x \sim \eta^{-1}$

$$\longrightarrow \quad U(g_0) = \frac{C}{\eta}, \quad C \sim 1 \quad \longrightarrow \quad P_2 = \frac{C}{\pi\nu} \sim 1$$

- $W < W_c$ delocalized phase: spontaneous symmetry breaking

manifold of saddle points

$$g_0(\Psi) \longrightarrow g_{0T}(\Psi) = g_0(\hat{T}\Psi) = g_0(\Psi^\dagger \bar{\hat{T}}\hat{T}\Psi,\ \Psi^\dagger\hat{\Lambda}\Psi) \qquad \bar{\hat{T}}\hat{\Lambda}\hat{T} = \hat{\Lambda}$$

$$\langle G_R(j,j)G_A(j,j)\rangle = \int Dg e^{-N\mathcal{L}(g)}U(g) = \int d\mu(\hat{T})\, U(g_{0T})\, e^{-\frac{\pi}{2}N\eta\nu \mathrm{Str}\left[\bar{\hat{T}}\hat{T}\right]}$$

$$P_2 = \frac{1}{\pi\nu}\lim_{\eta\to 0}\eta\,\langle G_R(j,j)G_A(j,j)\rangle = \frac{12}{N}\frac{g_{0,xx}^{(m+1)}}{\pi^2\nu^2} = \frac{3}{N}\frac{\langle\nu^2\rangle_{\mathrm{BL}}}{\nu^2} \qquad N \gg N_\xi$$

Near the transition: $\langle\nu^2\rangle_{\mathrm{BL}}/\nu^2 = N_\xi \gg 1$ — correlation volume $\quad P_2 = 3\frac{N_\xi}{N}$

Exact relations between RRG and infinite BL problems !

Generalized to correlation functions at arbitrary distance r
and of different eigenstates (energy separation ω)

Wave function correlations: Single wave function

RRG: $\alpha(r) = \langle |\psi_k^2(i)\psi_k^2(j)| \rangle$ $\quad r$ – distance between i and j

large $N \longrightarrow$ expressed in terms of infinite Bethe lattice correlation functions:

$$K_1(r) = \langle G_R(i,i)G_A(j,j)\rangle_{\rm BL} = \langle \tfrac{1}{16}(\Psi_{i,1}^\dagger \hat{K}\Psi_{i,1})(\Psi_{j,2}^\dagger \hat{K}\Psi_{j,2})\rangle_{\rm BL}$$

$$K_2(r) = \langle G_R(i,j)G_A(j,i)\rangle_{\rm BL} = \langle \tfrac{1}{16}(\Psi_{j,1}^\dagger \hat{K}\Psi_{i,1})(\Psi_{i,2}^\dagger \hat{K}\Psi_{j,2})\rangle_{\rm BL}$$

- Localized phase:

$$\alpha(r) = \frac{1}{\pi\nu N}\lim_{\eta\to 0}\eta K_1(r,\eta) \sim \frac{1}{N} m^{-r} e^{-r/\zeta} r^{-3/2}$$

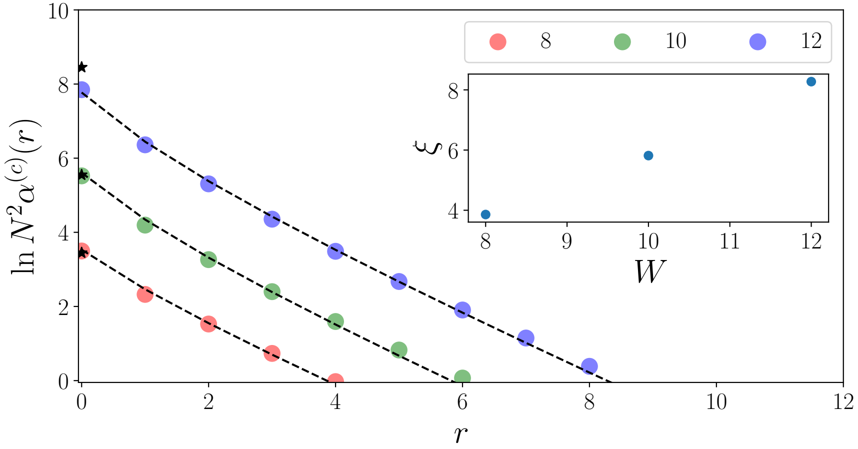
ζ – localization length

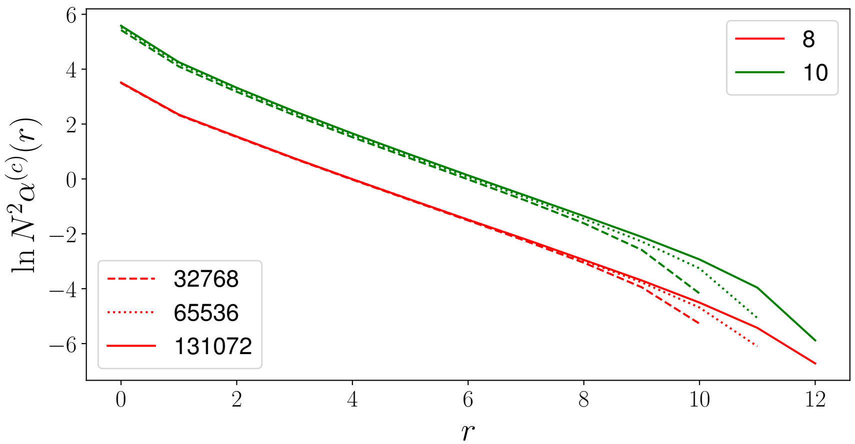
- Critical point: $\zeta = \infty \longrightarrow \alpha(r) \sim \dfrac{1}{N}\dfrac{m^{-r}}{r^{3/2}}$

- Delocalized phase, $N \gg N_\xi$:

$$\alpha(r) = \frac{1}{2\pi^2 N^2}[K_1(r) + 2K_2(r)] \sim \frac{N_\xi}{N^2}\frac{m^{-r}}{r^{3/2}}$$

$(r < \xi)$





Wave function correlations: Different wave functions

RRG: $\beta(r,\omega) = \langle|\psi_k^2(i)\psi_l^2(j)|\rangle$ $\qquad \omega = \epsilon_k - \epsilon_l \qquad r = \text{distance}(i,j)$

$$\beta(r,\omega) = \frac{1}{2\pi^2 N^2}\,\text{Re}\,K_1(r,\omega)$$

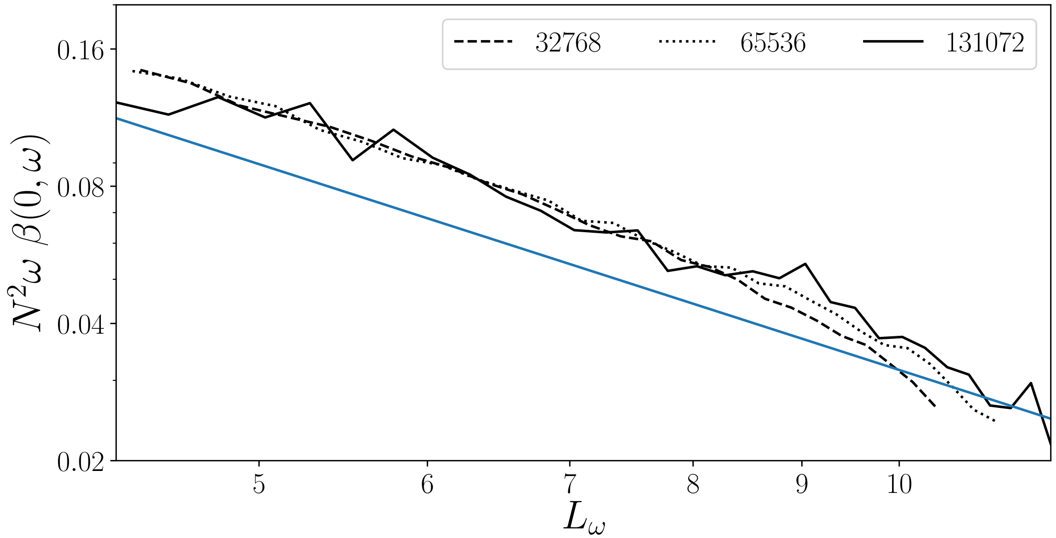
consider first $r = 0$

- **Critical point**

$$K_1(r=0,\omega=2i\eta) \simeq \frac{c_1^{(K)}}{\eta} + \frac{c_2^{(K)}}{\eta\ln^{\mu}1/\eta}$$

$$\longrightarrow \quad \beta(0,\omega) \sim \frac{1}{N^2\omega\ln^{\mu+1}1/\omega}$$

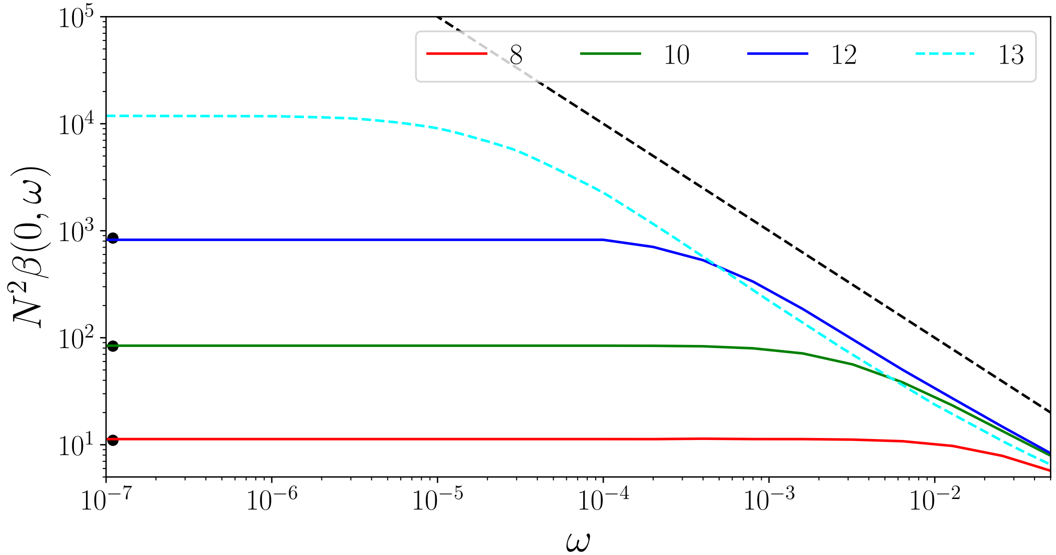
$\mu = 1/2$ **from ED**
and numerical solution of SC equation



- **Delocalized phase**

$$\beta(0,\omega) \sim \begin{cases} N_\xi/N^2, & \omega < \omega_\xi \\ \dfrac{1}{N^2\omega\ln^{\mu+1}1/\omega}, & \omega > \omega_\xi \end{cases}$$

$\omega_\xi \sim N_\xi^{-1}$ **(with log correction)**



Wave function correlations: $r - \omega$ plane

$$\beta(r,\omega) = \langle |\psi_k^2(i)\psi_l^2(j)| \rangle \qquad \omega = \epsilon_k - \epsilon_l \qquad r = \text{distance}(i,j)$$

$$\beta(r,\omega) = \frac{1}{2\pi^2 N^2}\,\mathrm{Re}\,K_1(r,\omega)$$

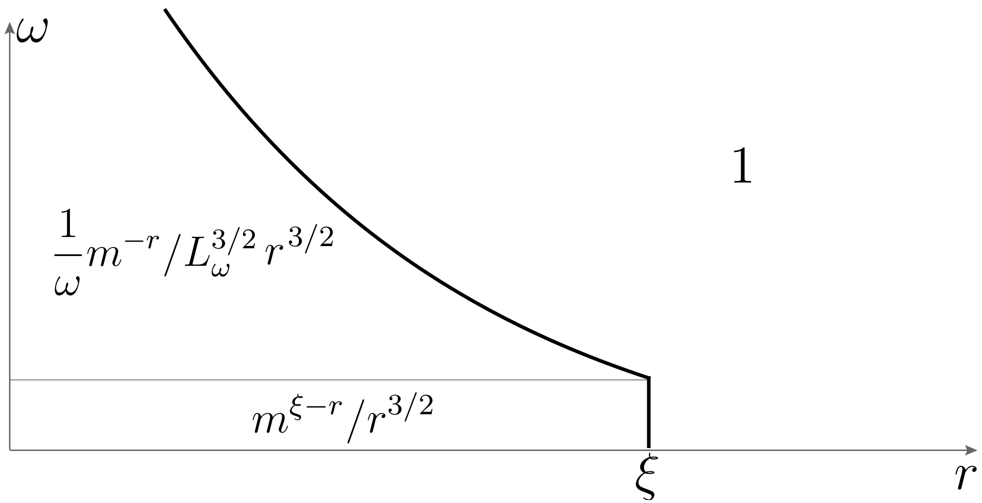
consider $W < W_c$

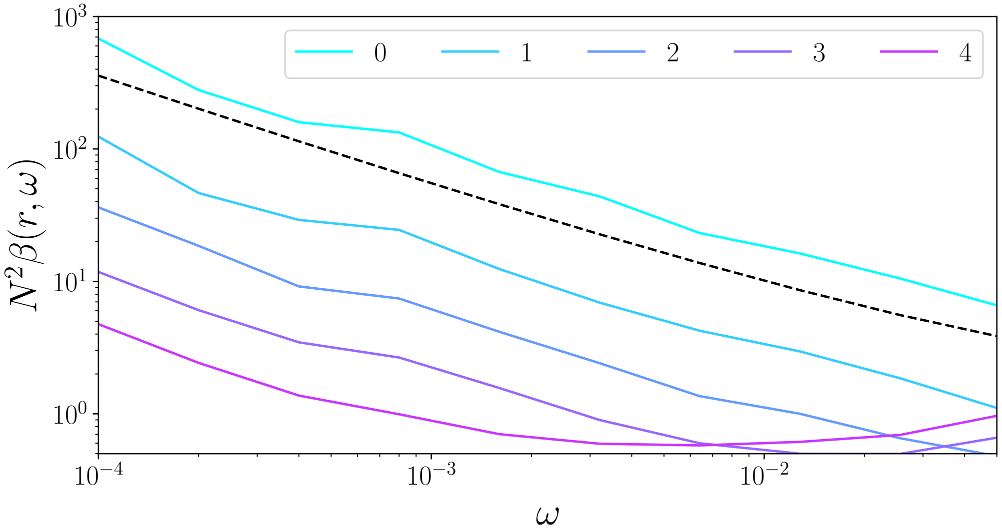
$$\beta(r,\omega) \sim \begin{cases} \dfrac{m^{\xi-r}}{N^2 r^{3/2}}, & r < \xi < L_\omega \\ \dfrac{m^{-r}}{N^2 \omega L_\omega^{3/2} r^{3/2}}, & r < L_\omega < \xi \end{cases}$$

"metallic" regime ($r < \xi < L_\omega$)

critical regime ($r < L_\omega < \xi$)

characteristic length scales: $\xi \sim (W_c - W)^{-1/2}$ $\qquad L_\omega = \log_m(1/\omega)$





Further dynamical observables: return probability, spectral statistics

Critical behavior

Correlation volume N_ξ $\longrightarrow$ correlation length ξ

Critical behavior: $\xi \sim (W_c - W)^{-\nu_{\rm del}}$ critical index $\nu_{\rm del} =?$

Self-consistency equation $\longrightarrow$ $m\lambda_\beta = 1$

λ_β – largest eigenvalue of certain integral operator

$\lambda_\beta(W) \simeq \frac{1}{2} - c_1\,(W - W_c) + c_2\left(\beta - \frac{1}{2}\right)^2$, has minimum at $\beta = 1/2$

Localized phase, $W > W_c$: β real

Critical point, $W = W_c$: $m\lambda_{1/2} = 1$ Abou-Chacra et al, 1973

Delocalized phase, $W < W_c$: spontaneous symmetry breaking

β becomes complex: $\beta = \frac{1}{2} \pm i\sigma\,,$ $\sigma \simeq \sqrt{\frac{c_1}{c_2}}(W_c - W)^{1/2}$

Correlation length $\ln N_\xi \simeq \dfrac{\pi}{\sigma}$ $\longrightarrow$ critical index $\boxed{\nu_{\rm del} = 1/2}$

$m = 2 \longrightarrow c_1 \simeq 1.59, \quad c_2 \simeq 0.0154 \longrightarrow \ln N_\xi \simeq 31.9\,(W_c - W)^{-1/2}$

ADM, Fyodorov, 1991, Tikhonov, ADM, 2019

Critical behavior

Numerical verification of $\nu_{\rm del} = 1/2$?

Kravtsov, Altshuler, Ioffe, Ann Phys 2018 found $\nu_{\rm del} \approx 1$. Contradiction?

We want an accurate determination of W_c and $\nu_{\rm del}$

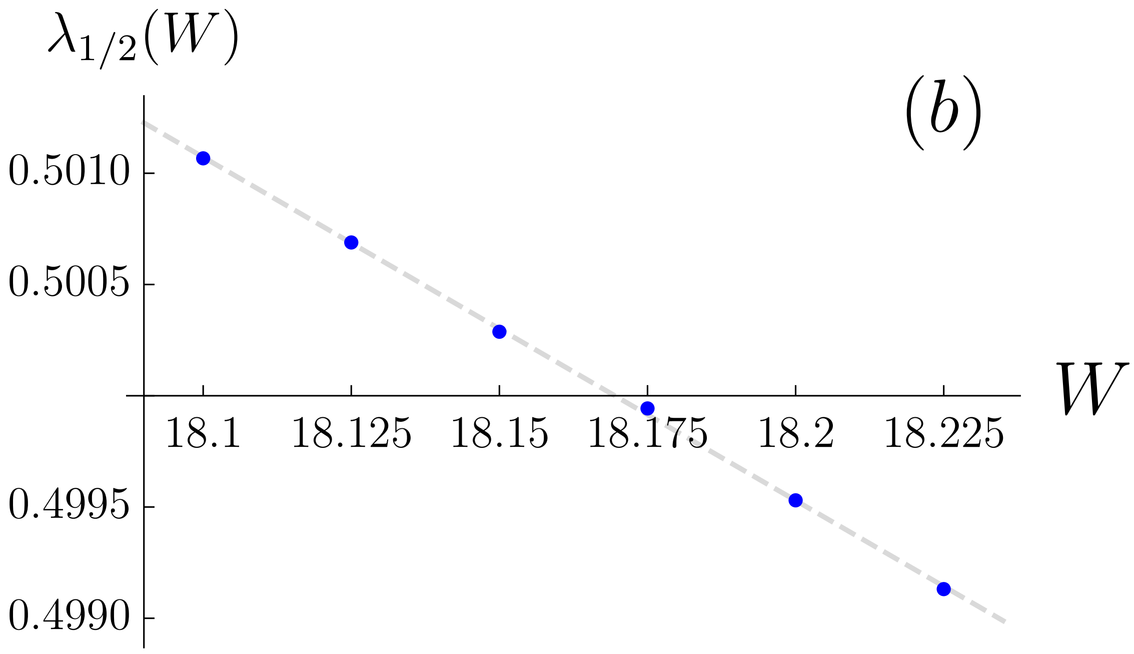
Exact diagonalization for RRG: system sizes not sufficient for this purpose

To approach much closer to the critical point, we use field theory and solve numerically the self-consistency equation

First step: accurate determination of W_c from the equation $m\lambda_{1/2} = 1$

$m = 2$

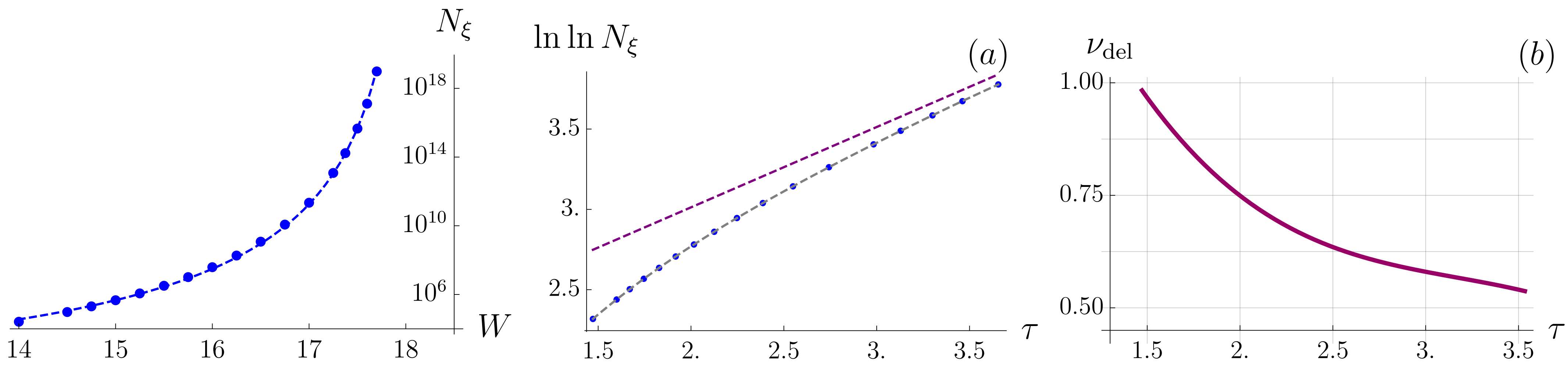
$$W_c = 18.17 \pm 0.01$$



Critical behavior: Numerical confirmation of $\nu_{\rm del} = 1/2$

Solve self-consistency equation by pool method (population dynamics) and thus determine N_ξ

$$\ln N_\xi \sim (W_c - W)^{-\nu_{\rm del}} \longrightarrow \frac{\partial \ln \ln N_\xi}{\partial \ln \tau} = \nu_{\rm del} \qquad \tau = -\ln(1 - W/W_c)$$



$m = 2 \longrightarrow$ **asymptotics** $\ln N_\xi = 31.9\,(W_c - W)^{-1/2}$

$$\boxed{\nu_{\rm del} = 1/2}$$

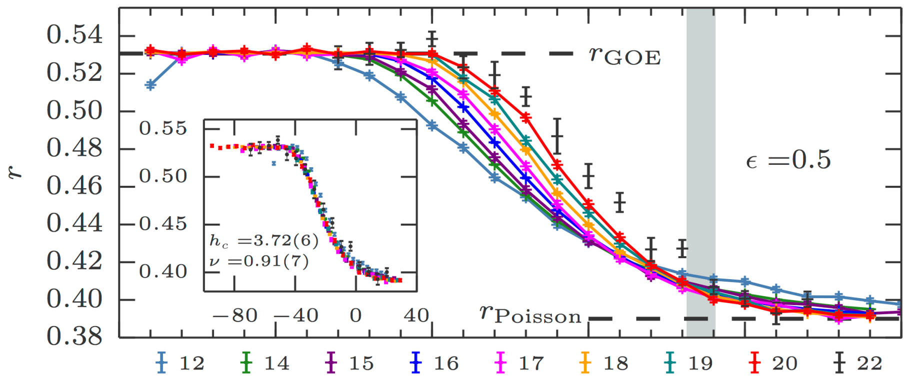
MBL with short-range interaction: Analogies to RRG

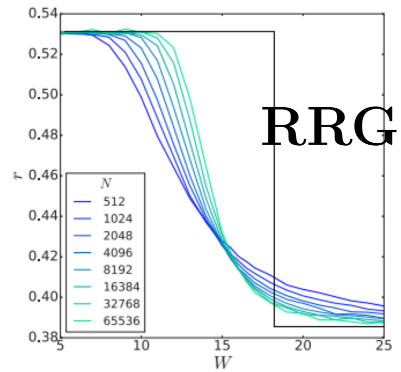
MBL with short-range interaction: XXZ spin chain in random field

Luitz, Laflorencie, Alet, PRB (2015); Mace, Alet, Laflorencie, arxiv:1812.10283

Striking similarities to RRG

- strong drift of crossing point
- critical point similar to localized phase



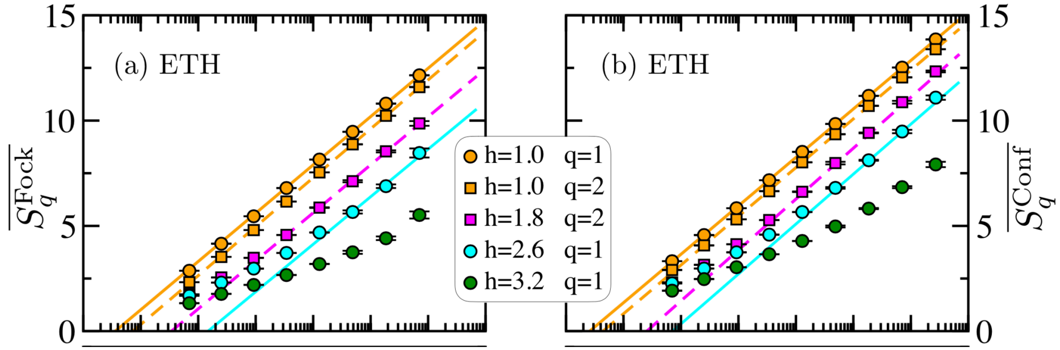


- ergodicity of the delocalized phase
- asymmetry of the critical behavior:

$\nu_{\text{del}} \simeq 0.45$ and $\nu_{\text{loc}} \simeq 0.76$

to be compared to

$\nu_{\text{del}} = 1/2$ and $\nu_{\text{loc}} = 1$ (RRG)



Numerically found exponents for MBL are close to those for RRG and strongly violate Harris criterion. Apparently, studied MBL systems are too small to exhibit asymptotic critical behavior. Intermediate, RRG-like fixed point – ?

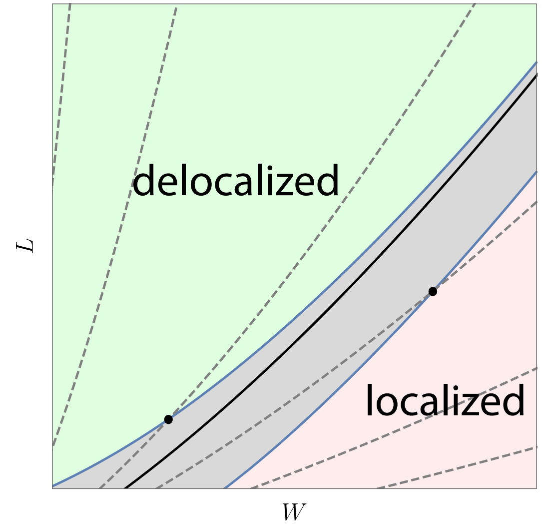
MBL with long-range interaction and RRG

Random spin chain with $1/r^{\alpha}$ interaction, $d < \alpha < 2d$

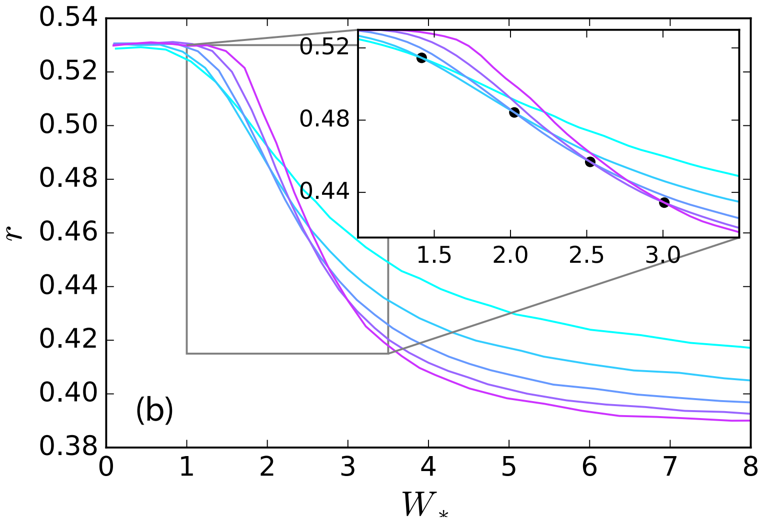
Mapping to RRG $\longrightarrow$ $W_c \sim L^{2d-\alpha} \ln L$

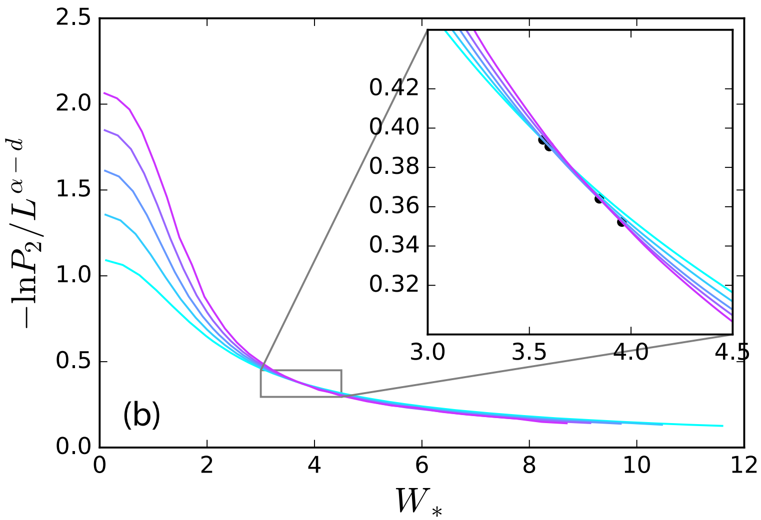
Agreement with exact diagonalization

$d = 1\,, \qquad \alpha = 3/2$



- Scaling of transition point
- Delocalized side: Ergodicity
- Critical point $\longrightarrow$ drift towards larger $W_* = W/L^{1/2} \ln L$





Summary

- **Localization transition on RRG. Approaches: (i) exact diagonalization, (ii) analytics, (iii) analytics + population dynamics. Full agreement.**
- **Ergodicity of the delocalized phase $W < W_c$, achieved for $N \gg N_\xi(W)$ with $\ln N_\xi \propto (W_c - W)^{-1/2}$**
- **Critical regime (of nearly localized character) for $N \ll N_\xi(W)$ $\longrightarrow$ peculiar crossover from criticality to ergodicity**
- **Detailed understanding of eigenfunction fluctuations and correlations, and level statistics.**
- **RRG as a very intricate $d = \infty$ limit of Anderson localization in d dimensions**
- **Index $\nu_{\text{del}} = 1/2$ confirmed numerically. Large corrections to scaling. Accurate evaluation of $W_c = 18.17 \pm 0.01$ (for $m = 2$) and of N_ξ up to 10^{19}**
- **RRG as a toy-model of MBL. Quantitative connections to long-range MBL. Strong qualitative analogies with short-range MBL.**