Diving into the depths of theoretical physics

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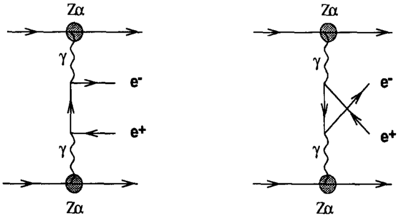
Dedicated to the 100th anniversary
of Professor I.M. Khalatnikov

Layout:

- e^+e^- pair production in collision of ultra-relativistic nuclei, Coulomb and unitarity corrections
- Spin filtering in storage rings.
- Polarization effects in non-relativistic $e^+\bar{p}$ scattering.
- Quasiclassical approximation for the amplitudes of basic processes in the field of heavy nuclei at high energies

$Z_A Z_B \to Z_A Z_B\, e^+ e^-$ in the Born approximation



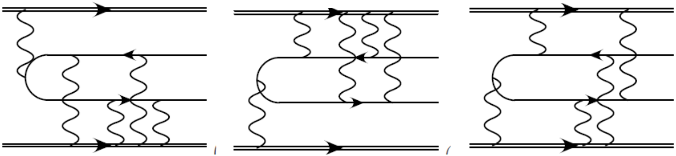
L.D. Landau, E.M. Lifshitz (1934), G. Racah (1937):

$$\sigma_{\mathrm{Born}} = \frac{28}{27\pi}\,\frac{\zeta}{m_e^2}\left[L^3 - 2.198\,L^2 + 3.821\,L - 1.636\right] ,$$

$$\zeta = (Z_A\alpha)^2 (Z_B\alpha)^2\,, \quad L = \ln(\gamma_A\gamma_B)\,,$$

$\alpha = 1/137$, $Z_{A,B}$ are the charge numbers of the nuclei A and B and $\gamma_{A,B}$ are their Lorentz factors.

$Z_A Z_B \to Z_A Z_B\, e^+ e^-$, Coulomb corrections



Experimental results of SPS at CERN ($\gamma_A = 100$, $\gamma_B = 1$):
there are no Coulomb corrections though $Z_A\alpha$, $Z_B\alpha \sim 1$!

B.Segev, J.C.Wells(1998);
A.J.Baltz, L.McLerran (1998);
U.Eichmann, J.Reinhardt, S.Schramm, W.Greiner (1999):
The exact in $Z_{A,B}\alpha$ cross section coincides with that calculated in the Born approximation! Contradiction with the Weizsäcker-Williams approximation.

D.Yu.Ivanov, A.Schiller, and V.G. Serbo (1999), R.N. Lee and A.I. Milstein (2000,2001):

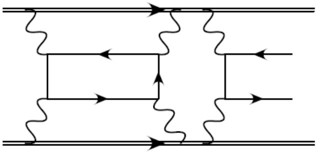
$$\sigma^C = -\frac{28}{9\pi}\frac{\zeta}{m_e^2}\,L^2\left[f(Z_A\alpha)+f(Z_B\alpha)\right],$$

$$\sigma^{CC} = \frac{56}{9\pi}\frac{\zeta}{m_e^2}\,L\,f(Z_A\alpha)f(Z_B\alpha),$$

$$f(x) = \mathrm{Re}\,\psi(1+ix)+C = x^2\sum_{n=1}^{\infty}\frac{1}{n(n+x^2)}\,.$$

Agreement of σ^C with the Weizsäcker-Williams approximation.

$Z_A Z_B \to Z_A Z_B\, e^+ e^-$, unitarity corrections



R.N. Lee, A.I. Milstein, V.G. Serbo (2002)

$$\sigma_{unit} = -\frac{2.66\zeta^2 L^2}{m_e^2} \text{ for } Z_A\alpha,\, Z_B\alpha \ll 1\,, \zeta L \lesssim 1$$

Explanation of the paradox [R.N.Lee, A.I.Milstein (2000)].

The cross section is proportional to J,

$$J = \int q_\perp^2 |F|^2 d\Omega \,, \; F = \frac{ip}{2\pi} \int d^2\rho \, \mathrm{e}^{-i\boldsymbol{q}_\perp \boldsymbol{\rho}} \left(1 - \mathrm{e}^{-i\chi(\rho)}\right) ,$$

$$J = \int \frac{d^2 q_\perp}{(2\pi)^2} \iint d^2\rho d^2\rho' \, \mathrm{e}^{i\boldsymbol{q}_\perp(\boldsymbol{\rho}' - \boldsymbol{\rho})} \, \mathrm{e}^{-i(\chi(\rho) - \chi(\rho'))} \boldsymbol{\nabla}_\rho \chi(\rho) \boldsymbol{\nabla}_{\rho'} \chi(\rho') \,.$$

$$J_B = \int d^2\rho \, [\boldsymbol{\nabla}_\rho \chi(\rho)]^2 \,, \quad \chi(\rho) = \int_{-\infty}^{+\infty} V(\rho, z) \, dz$$

If we change the order of integration and use the relation $\int d^2 q \exp[i\boldsymbol{q}_\perp(\boldsymbol{\rho} - \boldsymbol{\rho}')] = (2\pi)^2 \delta(\boldsymbol{\rho} - \boldsymbol{\rho}')$, we come to the wrong conclusion that $J = J_B$!

In fact

$$R = J - J_B = -8\pi(Z\alpha)^2 f(Z\alpha)\,, \quad f(x) = \mathrm{Re}\,\psi(1+ix) + C\,.$$

This result is independent of screening radius though $|F|^2 = |F_B|^2$ for $q_\perp \gg r_{scr}^{-1}$, where r_{scr} is the screening radius!

R appears in:

- The Coulomb corrections to the cross section of e^+e^- pair production in ultrarelativistic heavy-ion collisions;
- The Coulomb corrections in the Moliére's formula for multiple scattering ;
- The Coulomb corrections to the spectrum of bremsstrahlung (though the Coulomb corrections to the differential cross section of bremsstrahlung are very sensitive to screening!).

It is not legal to change the order of integration over $\boldsymbol{q}_\perp$ and ρ !
If one restrict the region of integration over $\boldsymbol{q}_\perp$ by the condition $\boldsymbol{q}_\perp < Q$, then

$$R = \lim_{Q\to\infty} \frac{Q}{2\pi} \int\int d\boldsymbol{\rho} d\boldsymbol{\rho}' \frac{J_1(Q|\boldsymbol{\rho}-\boldsymbol{\rho}'|)}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|} \times \left\{\exp[i\chi(\rho) - i\chi(\rho')] - 1\right\} \boldsymbol{\nabla}_\rho \chi(\rho) \boldsymbol{\nabla}_{\rho'} \chi(\rho') \, .$$

After the substitution $\boldsymbol{\rho} \to \boldsymbol{\rho}/Q$ we can pass to the limit $Q \to \infty$ in the integrand using the asymptotics $V(\boldsymbol{r}) \to -Z\alpha/r$ and $\chi(\boldsymbol{\rho}) \to 2Z\alpha(\ln\rho + const)$ at $r \to 0$.
Then we obtain the final universal result!

What to do with the experimental results?!

Explanation: contribution of next-to-leading logarithmic correction to the Coulomb corrections ($\propto L = \ln(\gamma_A \gamma_B)$), R.N.Lee, A.I.Milstein (2009):

$$\sigma_A^C = -\frac{28}{9\pi}\frac{\zeta}{m_e^2} f(Z_A\alpha)\left\{L^2 + \left[G(Z_A\alpha) + \frac{20}{21}\right]L\right\}$$

$$G(Z_A\alpha) = 2\int_{2m_e}^{\infty}\frac{d\omega}{\omega}\left[\frac{\sigma_{\gamma A}^C(\omega)}{\sigma_{\gamma A}^C(\infty)} - 1\right].$$

The quantity $\sigma_{\gamma A}^C(\omega)$ is the Coulomb corrections to the cross section of e^+e^- pair production by real photon in the Coulomb field, and

$$\sigma_{\gamma A}^C(\infty) = -\frac{28\alpha(Z_A\alpha)^2}{9m_e^2} f(Z_A\alpha)$$

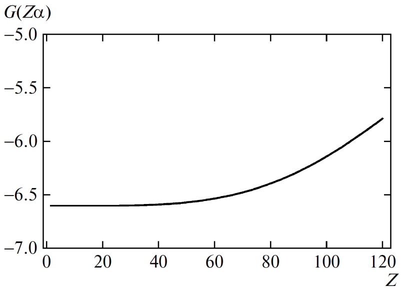
The function $G(Z\alpha)$ is huge! At $m/\omega \ll 1$ the correction $\delta\sigma^{C}_{\gamma A}(\omega)$ over $m/\omega \ll 1$ reads [R.N.Lee, A.I.Milstein, V.M.Strakhovenko (2004)]:

$$\delta\sigma^{C}_{\gamma A}(\omega) = -\frac{28\alpha(Z_A\alpha)^2}{9m_e^2}\left[\frac{\pi^4}{2}\mathrm{Im}\, g(Z\alpha) + 4\pi(Z\alpha)^3 f_1(Z\alpha)\right]\frac{m_e}{\omega}\,,$$

$$g(Z\alpha) = Z\alpha\frac{\Gamma(1-iZ\alpha)\Gamma(1/2+iZ\alpha)}{\Gamma(1+iZ\alpha)\Gamma(1/2-iZ\alpha)}\,.$$

The function $f_1(Z\alpha) \lesssim 1$ is related to the total cross section σ_{bf} of the so-called bound-free pair production when an electron is produced in a bound state,

$$\sigma_{bf}(Z\alpha) = \frac{4\pi\alpha\,(Z\alpha)^5}{m_e^2}\,f_1(Z\alpha)\,\frac{m}{\omega}\quad,$$



As a result we have almost total numerical cancellation between the term $\propto L^2$ and $\propto L$ in the Coulomb corrections.

So, the agreement between a few theories does not guarantee the correct result!

The agreement between theory and experiment does not guarantee the correct physical conclusions!

Spin filtering in storage rings

Task: to obtain polarized antiproton beam.

We have the usual equation

$$\frac{d}{dt}\int d\boldsymbol{r}\,\Psi_{\boldsymbol{k}}^{+}(\boldsymbol{r})\mathcal{O}_H\Psi_{\boldsymbol{k}}(\boldsymbol{r}) = i\int d\boldsymbol{r}\,\Psi_{\boldsymbol{k}}^{+}(\boldsymbol{r})[H,\mathcal{O}_H]\Psi_{\boldsymbol{k}}(\boldsymbol{r})\,,$$

$$\Psi_{\boldsymbol{k}}(\boldsymbol{r}) = \frac{1}{\sqrt{V}}e^{-\lambda r}\,\psi_{\boldsymbol{k}}(\boldsymbol{r})\,,\;\; H\psi_{\boldsymbol{k}}(\boldsymbol{r}) = E\psi_{\boldsymbol{k}}(\boldsymbol{r})\,,\;\; \mathcal{O}_H = e^{iHt}\mathcal{O}e^{-iHt}\,,$$

where H is the Hamiltonian of the system

$$\frac{d}{dt}\int d\boldsymbol{r}\,\Psi_{\boldsymbol{k}}^{+}(\boldsymbol{r})\mathcal{O}_H\Psi_{\boldsymbol{k}}(\boldsymbol{r})$$

$$= iN\int d\boldsymbol{r}\,\psi_{\boldsymbol{k}}^{+}(\boldsymbol{r})\left\{[e^{-\lambda r},H]\mathcal{O}_H e^{-\lambda r} + e^{-\lambda r}\mathcal{O}_H[e^{-\lambda r},H]\right\}\psi_{\boldsymbol{k}}(\boldsymbol{r})\,,$$

where $N = 1/V$ is the density. Since $[e^{-\lambda r},H] \propto \lambda$, only contribution of large distances compensate small λ !

At large distances

$$\psi_{\boldsymbol{k}}(\boldsymbol{r}) = \left[e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \frac{e^{ikr}}{r}\,F\right]\chi_1\chi_2\,,$$

where $\boldsymbol{k}$ is the initial momentum, $\chi_{1,2}$ are the spin wave functions, F depends on $\boldsymbol{n}_0 = \boldsymbol{k}/k$, $\boldsymbol{n} = \boldsymbol{r}/r$, and the spin operators. Finally, we obtain the kinetic equation [L.C.Balling, R.J.Hanson, and F.M.Pipkin (1964); V.G.Baryshevsky, A.G.Shekhtman (1996); N.N.Nikolaev, F.F.Pavlov (2006); A.I.Milstein, S.G.Salnikov (2013)]:

$$\frac{d}{dt}\,\langle\mathcal{O}\rangle = vN\,\mathrm{Sp}\left\{\rho(t)\left[\int d\Omega_{\boldsymbol{n}} F^{+}\mathcal{O}F - \frac{2\pi i}{k}\Big(F^{+}(0)\mathcal{O} - \mathcal{O}F(0)\Big)\right]\right\}.$$

Here $v = k/M$, $d\Omega_n$ is the differential of the solid angle corresponding to vector $\boldsymbol{n}$, $F(0)$ is F at $\boldsymbol{n} = \boldsymbol{n}_0$, and $\rho(t)$ is the density matrix, the trace is taken over the spin indexes.

If the target polarization vector $\boldsymbol{P}_T$ is parallel or perpendicular to the antiproton beam axis $\boldsymbol{\nu}$, then kinetic of polarization becomes essentially simpler [A.I.Milstein, V.M.Strakhovenko (2005); N.N.Nikolaev, F.F.Pavlov (2006)]. In these cases antiproton beam polarization vector $\boldsymbol{P}_B \parallel \boldsymbol{P}_T$.

The cross section σ of $p\bar{p}$ interaction reads

$$\sigma = \sigma_{\mathrm{el}}(p\bar{p} \to p\bar{p}) + \sigma_{\mathrm{cex}}(p\bar{p} \to n\bar{n}) + \sigma_{\mathrm{ann}}(p\bar{p} \to \mathrm{mesons}).$$

All cross sections are summed up over final spin states, σ_{el} includes pure Coulomb cross section, hadronic cross section, and interference term, which can't be omitted. Noticeable polarization can be obtained only if some antiprotons are dropped out of the beam!

Spin-dependent cross section can be written in the form

$$\sigma = \sigma_0 + (\boldsymbol{\zeta}_1 \cdot \boldsymbol{\zeta}_2)\sigma_1 + (\boldsymbol{\zeta}_1 \cdot \boldsymbol{\nu})(\boldsymbol{\zeta}_2 \cdot \boldsymbol{\nu})(\sigma_2 - \sigma_1), \qquad (1)$$

where $\boldsymbol{\zeta}_{1,2}$ are unit vectors collinear to the particles spins. For the antiproton beam polarization $P_B(t)$ and the number of particles in the beam $N(t)$, we have

$$P_B(t) = \tanh\left[\frac{t}{2}\left(\Omega_-^{out} - \Omega_+^{out}\right)\right],$$
$$N(t) = \frac{1}{2}N(0)\left[\exp\left(-\Omega_+^{out}t\right) + \exp\left(-\Omega_-^{out}t\right)\right],$$
$$\Omega_\pm^{out} = nf\left\{\sigma_0 \pm P_T\left[\sigma_1 + (\boldsymbol{\zeta}_T \cdot \boldsymbol{\nu})^2(\sigma_2 - \sigma_1)\right]\right\},$$

where $\boldsymbol{\zeta}_T = \boldsymbol{P}_T/P_T$, n is the areal density of the target and f is the beam revolving frequency.

First attempt of the PAX (Polarized Antiproton eXperiment) collaboration:
Effect of C.J.Horowitz and H.O.Meyer (1994).
This effect was based on the wrong description of the interaction of stored antiprotons with a polarized target [A.I.Milstein, V.M.Strakhovenko (2005)].
The project was closed.
To predict $\sigma_{1,2}$, it is possible to use optical potentials

$$V_{N\overline{N}} = U_{N\overline{N}} - iW_{N\overline{N}},$$

where $W_{N\overline{N}}$ describes annihilation into mesons. There are too many potentials with different predictions! Another possibility is to measure $\sigma_{1,2}$. It is not done till now. So, now it is not clear whether it is possible to use filtering!

Polarization effects in non-relativistic e^+p scattering

Second attempt of the PAX collaboration:
Effect of H.Arenhövel and Th.Walcher (2007).
This effect was based on the wrong description of the interaction of stored antiprotons with a polarized positron beam at low relative velocity.

[A.I.Milstein, S.G.Salnikov, V.M.Strakhovenko (2008)].
The project was closed.
Brilliant idea to use huge amplification of the cross section by a factor $C^2(\xi)$, where the famous SGS factor $C(\xi)$ for e^-p or $e^+\bar{p}$ interaction reads:

$$C(\xi) = \frac{2\pi\xi}{1-\exp(-2\pi\xi)}\,, \quad \xi = \alpha/v\,,$$

where v is the relative velocity.

H.Arenhövel and Th.Walcher estimated the radial integrals numerically. They concluded that antiprotons can be easily polarized!
The polarization degree of the antiproton beam , $P(t)$, is

$$P(t) = P_e P_0 \left(1 - \mathrm{e}^{-\Omega t}\right) \ , \quad P_0 = \frac{\sigma_{+-} - \sigma_{-+}}{\sigma_{+-} + \sigma_{-+}} \ ,$$

$$\Omega = f n l \sigma_{tot} \frac{v}{V_b} \ , \qquad \sigma_{tot} = \sigma_{+-} + \sigma_{-+} \ .$$

f is a revolution frequency, n is a density of the positron beam, l is the length of the interaction region, V_b is the antiproton beam velocity, P_e is the polarization degree of the positron beam, σ_{-+} and σ_{+-} correspond to spin flip from $\zeta_p = \zeta_e$ and $\zeta_p = -\zeta_e$, respectively.

We obtain

$$P_0 = \frac{(2 - \ln 2)}{3 - 2\ln 2 + \ln(l_{max}^2/\xi^2)/(2\pi\xi)^2}\,,$$

$$\sigma_{tot} = \sigma_0 \left\{ (2\pi\xi)^2 (3 - 2\ln 2) + \ln \frac{l_{max}^2}{\xi^2} \right\},$$

$l_{max} \sim mv\rho \gg \xi$, where $\rho \sim$ transverse beam size. At v=0.002 and $\rho \approx 1$ cm , we have $P_0 \approx 0.78$ which is big enough. However, $\sigma_{tot} \approx 0.75\,\mathrm{mb}$ that drastically differs from the result $\sigma_{tot} \approx 4 \cdot 10^{+13}$ barn obtained by WA. When the correct value of σ_{tot} is used, the beam polarization time, $\tau = \Omega^{-1}$, becomes enormously large for the parameters of positron beams available now!

Quasiclassical approximation

For small scattering angles ϑ angular momenta l are large

$$l \sim \varepsilon\varrho \sim \frac{\varepsilon}{q} \sim \frac{1}{\vartheta} \gg 1\,,$$

ϱ is the impact parameter, q is the momentum transfer, and ε is the electron energy. One can use the quasiclassical wave function and Green's function for electron in the external potential $V(\boldsymbol{r})$. In the leading quasiclassical approximation: R.N.Lee, A.I.Milstein, V.M.Strakhovenko (2000). First quasiclassical corrections (the parameter of expansion is $1/l \ll 1$): P.A.Krachkov, A.I.Milstein (2015). For the superposition of the potential $V(\boldsymbol{r})$ and the laser field: A.Di Piazza, A.I.Milstein (2014).

In the leading approximation:

$$\psi_P^{(\pm)}(\boldsymbol{r}) = \pm \int \frac{d^2q}{i\pi} \exp\left[i\boldsymbol{p}\cdot\boldsymbol{r} \pm iq^2 \mp i\int_0^\infty dx V(\boldsymbol{r}_x)\right]$$

$$\times \left\{1 \mp \frac{1}{2p}\int\limits_0^\infty dx\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}V(\boldsymbol{r}_x)\right\}u_P\ ,\ \boldsymbol{r}_x = \boldsymbol{r} \mp x\boldsymbol{n} + \frac{\boldsymbol{q}}{p}\sqrt{2|\boldsymbol{r}\cdot\boldsymbol{n}|}\,,$$

$\boldsymbol{q}$ is a two-dimensional vector, $\boldsymbol{n} = \boldsymbol{p}/p$, $\boldsymbol{q}\cdot\boldsymbol{n} = 0$,
u_P is the conventional Dirac spinor, $P = (\varepsilon_p, \boldsymbol{p})$.
Integration over $\boldsymbol{q}$ corresponds to account for quantum fluctuations.
For the Coulomb field $V(r) = -Z\alpha/r$,
$\psi_P^{(\pm)}(\boldsymbol{r}) \Longrightarrow$ Furry-Sommerfeld-Maue wave function!

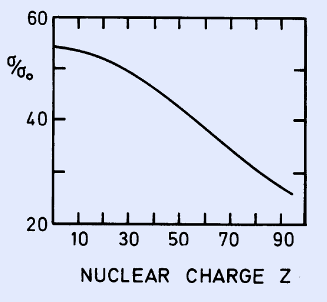
If one replace $\boldsymbol{r}_x$ by $\boldsymbol{R}_x = \boldsymbol{r} \mp x\boldsymbol{n}$, we obtain the conventional eikonal wave function

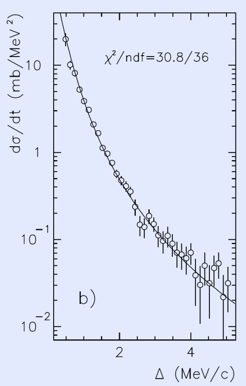
$$\psi^{(in,\,out)}_{P,\,eik}(\boldsymbol{r}) = \exp\left[i\boldsymbol{p}\cdot\boldsymbol{r} \mp i\int_0^\infty dx V(\boldsymbol{R}_x)\right]$$

$$\times\left\{1 \mp \frac{1}{2p}\int\limits_0^\infty dx\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}V(\boldsymbol{R}_x)\right\}u_P\,.$$

Using the quasiclassical wave functions, we investigate (exactly in the parameters of the atomic field) numerous high-energy QED processes (see P.A.Krachkov, R.N.Lee, and A.I.Milstein; Usp. Fiz. Nauk (2016)). Example: photon splitting in the atomic field. First successful observation at Budker Institute (2001).

Why it is interesting? The Coulomb corrections change drastically the results as compared to the Born result





Left figure: total cross section in units $\sigma_0 = (Z\alpha)^4 r_0^2/16\pi$, A.I.Milstein and V.M. Strakhovenko (1983)
Right figure: $d\sigma/dt$ as a function of the momentum transfer Δ for a molecule of bismuth germanate, data - circles, theory - solid line.

Conclusion

- Precision experiments and analysis of their results require a deep understanding of the basics of theoretical physics. Otherwise, you will discover New Physics every day. However, this physics will be " new " only for you.
- We should try to safe scientific traditions of Landau's school. This is important not only for our country but also for all scientific community.
- Many thanks to Professor Khalatnikov for his contribution to the right teaching of theoretical physicists in our country!

Thank you!