Odd Fluids

Alexander Abanov

Stony Brook University

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Tankut Can
CUNY, New York

Gustavo Monteiro
UNICAMP, Brazil

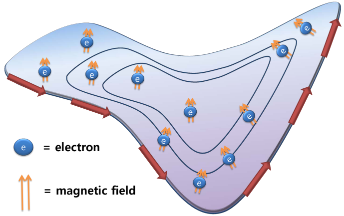
Sriram Ganeshan
CUNY, New York

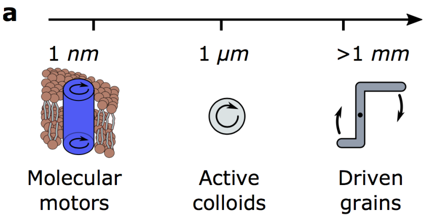
Motivation

- FQHE as exotic isotropic parity breaking fluid
- Isotropic fluids with broken parity in 2d
- Manifestation of odd/Hall viscosity in the bulk and on the boundary
- Variational and Hamiltonian formulation of fluid dynamics
- Role of topological terms in boundary conditions

Isotropic fluids with broken parity

- Rotating He-II, plasma in magnetic field, He3-A films
- Quantum Hall fluids (Gromov, AA '13-'15)
- Vortex fluids (Wiegmann, AA, '14)
- Chiral active fluids (Souslov, Banerjee, Vitelli, AA, '17)





Two-dimensional fluids!

Introduction: Hydrodynamics

- Separation of scales and emergence
- Local equilibration
- Symmetries and conservation laws, universality
- Gradient expansion
- Hydrodynamics

Role of symmetries: conservation laws + restrictions on constitutive relations

Remark: Macroscopic hydrodynamics

- Typically hydrodynamics is derived from microscopic theory, e.g., using kinetic equation
- However, one can also start with hydrodynamic equations and average them over even bigger scales
- This approach is known as “macroscopic hydrodynamics”
- Early example: Hall-Vinen-Bekarevich-Khalatnikov (HVBK) hydrodynamics (Hall, Vinen ’56, Hall, ’58, Mamaladze, Matinyan ’60, Bekarevich, Khalatnikov ’61 (review Sonin)
- Collection of vortices in rotating He-II is considered as an effective medium obeying HVBK hydrodynamics

Example: Barotropic fluid

Two conserved quantities ρ and p_i

$$\partial_t \rho + \partial_i j_i = 0 \qquad \text{mass conservation}$$
$$\partial_t p_i + \partial_j \Pi_{ij} = 0 \qquad \text{momentum conservation}$$

j_i - mass current, Π_{ij} momentum flux tensor, should be expressed in terms of ρ, p_i and gradients using symmetries (isotropy, Galilean invariance, ...).

$$\begin{aligned} j_i &= \rho v_i + \dots \\ p_i &= \rho v_i + \dots \qquad \text{constitutive relations} \\ \Pi_{ij} &= p_i v_j + p\delta_{ij} + \dots = p_i v_j - T_{ij} \\ p &= p(\rho) \end{aligned}$$

Zero-order barotropic fluid dynamics. Continuity and Euler equations for ρ and v_i.

Barotropic fluid: first order hydro

Continuity equations (no external forces):

$$\partial_t \rho + \partial_i j_i = 0\,,$$
$$\partial_t p_i + \partial_j (p_i v_j) = \partial_j T_{ij}\,.$$

Constitutive relations might have terms linear in gradients

$$j_i = \rho v_i + A\partial_i^* \rho\,, \qquad p_i = \rho v_i\,, \qquad p = p(\rho)\,,$$

$$T_{ij} = -p\delta_{ij} + \eta_e(\partial_i v_j + \partial_j v_i - \delta_{ij}\partial_k v_k) + \eta_b \delta_{ij}\partial_k v_k + G\omega\delta_{ij} + \eta_o(\partial_i v_j^* + \partial_i^* v_j)$$

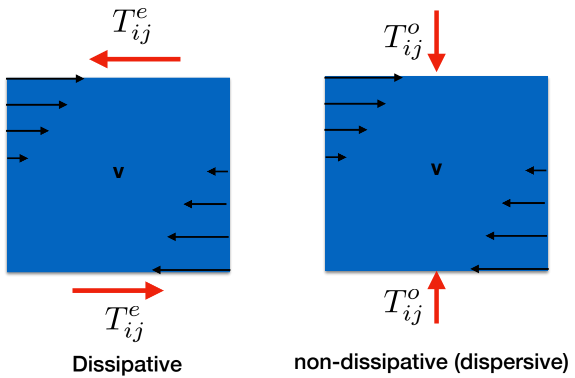
shear viscosity — bulk viscosity — odd pressure — odd viscosity

$a_i^* \equiv \epsilon^{ij} a_j$ – rotation by 90° clockwise – breaks parity!

Hall (odd) viscosity

$a_i^* \equiv \epsilon^{ij} a_j$ – rotation by 90° clockwise – breaks parity!

$$T^e_{ij} = \nu_e \rho(\partial_i v_j + \partial_j v_i) \qquad T^o_{ij} = \nu_o \rho(\partial_i v_j^* + \partial_i^* v_j)$$



Avron, Seiler, Zograf, 1995; Avron, 1998

Measuring Hall viscosity

TRANSVERSE MOMENTUM TRANSPORT IN VISCOUS FLOW OF DIATOMIC GASES IN A MAGNETIC FIELD

J. KORVING, H. HULSMAN, H. F. P. KNAAP and J. J. M. BEENAKKER
Kamerlingh Onnes Laboratory, Leiden, The Netherlands

Received 1 March 1966

It is proved experimentally that, in a magnetic field, transverse momentum transport occurs in viscous flow of diatomic molecules. The experimental results agree with theoretical calculations. The sign of the molecular gyromagnetic ratio can be determined.

Science REPORTS

Cite as: A. I. Berdyugin *et al.*, *Science* 10.1126/science.aau0685 (2019).

Measuring Hall viscosity of graphene's electron fluid

A. I. Berdyugin[1], S. G. Xu[1,2], F. M. D. Pellegrino[3,4], R. Krishna Kumar[1,2], A. Principi[1], I. Torre[5], M. Ben Shalom[1,2], T. Taniguchi[6], K. Watanabe[6], I. V. Grigorieva[1], M. Polini[1,7], A. K. Geim[1,2]*, D. A. Bandurin[1]*

nature physics ARTICLES
https://doi.org/10.1038/s41567-019-0603-8

The odd free surface flows of a colloidal chiral fluid

Vishal Soni[1,8], Ephraim S. Bililign[1,8], Sofia Magkiriadou[1,7,8], Stefano Sacanna[2], Denis Bartolo[3], Michael J. Shelley[4,5] and William T. M. Irvine[6]*

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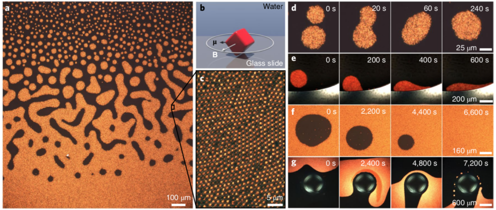

We consider fluid dynamics of

- two-dimensional fluid
- compressible
- isotropic
- parity breaking
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Goals: variational principle, Hamiltonian, boundary dynamics

Compressible hydro with ν_o

Continuity:

$$\partial_t \rho + \partial_i (\rho v_i) = 0$$

$$j_i = \rho v_i$$

Momentum conservation:

$$\partial_t (\rho v_i) + \partial_j (\rho v_i v_j) = \partial_j T_{ij}$$

$$p_i = \rho v_i$$

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Important: we identified mass density current and momentum density!

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Stress tensor:

$$T_{ij} = -p\delta_{ij} + \nu_o \rho(\partial_i^* v_j + \partial_i v_j^*) + \nu_e \rho(\partial_i v_j + \partial_j v_i)$$

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No dissipation:

$$T_{ij} = -p\delta_{ij} + \nu_o \rho(\partial_i^* v_j + \partial_i v_j^*)$$

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Boundary conditions at $y = h(x,t)$:

$$\left(\frac{\partial \Gamma}{\partial t} - \boldsymbol{v}\right)_n = 0\,, \qquad T_{ij} n_j\Big|_\Gamma = 0\,.$$

Incompressible limit: $\frac{dp}{d\rho} = c_s^2 \to \infty$.

Contents of the talk

- Preliminaries: fluid dynamics, variational principle, free surface etc.
- Fluid dynamics with odd viscosity and odd surface waves
- Variational principle and Hamiltonian formulation of hydro with odd viscosity
- Free surface

Hamiltonian structure of barotropic fluid

Poisson's brackets (notations $\rho = \rho(x)$, $\rho' = \rho(x')$ etc.)

$$\begin{aligned}\{\rho, \rho'\} &= 0 \\ \{\rho, v_i'\} &= \partial_i \delta(x - x') \\ \{v_i, v_j'\} &= -\frac{1}{\rho}(\partial_i v_j - \partial_j v_i)\delta(x - x')\end{aligned}$$

Hamiltonian generating equations $\partial_t q = \{H, q\}$:

$$H = \int dx \left[\frac{\rho v^2}{2} + \varepsilon(\rho)\right]$$

Equations of motion $(p = \rho\varepsilon_\rho - \varepsilon)$

$$\partial_t \rho + \partial_i(\rho v_i) = 0\,, \qquad \partial_t v_i + (v_j \partial_j) v_i = -\rho^{-1}\partial_i p\,.$$

Landau, '41; Dzyaloshinskii, Volovick, '79

Casimirs

Infinitely many integrals of motion in 2d hydro:

$$I_n = \int dx\, \rho \left(\frac{\omega}{\rho}\right)^n , \qquad n = 0, 1, 2, \ldots$$

Conserved for any Hamiltonian!

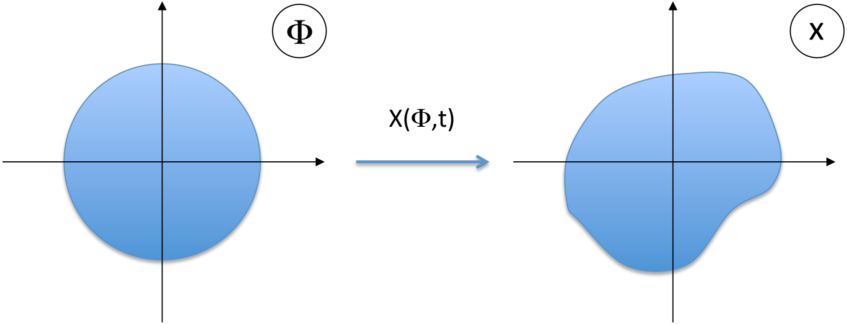
$$\{I_n, \rho\} = 0 \,, \qquad \{I_n, v_i\} = 0 \,.$$

Degeneracy of Poisson structure, not symmetry of the Hamiltonian. I_n — Casimirs. This degeneracy is an obstacle to getting variational principle.

$$\{p, q\} = \kappa \,, \quad H(p, q) \quad \longrightarrow \quad L(q, \dot{q}) = \frac{1}{\kappa} p\dot{q} - H(p, q)$$

Problem if κ is not invertible $\to$ symplectic (Hamiltonian) reduction.

From Lagrangian to Eulerian description



Two descriptions:

- Lagrangian: $x_i(\Phi_\alpha, t)$, $\dot{x}_i(\Phi_\alpha, t)$
- Eulerian: $v_k(x_i, t)$
- relation: $v_k(x_i, t) = \dot{x}_k(\Phi_\alpha, t)\Big|_{\Phi_\alpha = \Phi_\alpha(x_i, t)}$

Relabeling symmetry

$$\Phi_\alpha \to F_\beta(\Phi_\alpha)$$

Variational principle for fluid dynamics

Action for barotropic fluid dynamics:

$$S[\rho, \theta, \alpha, \beta, v_i] = -\int dt \int d\boldsymbol{x} \left[\rho \left(u_0 + u_i v^i - \frac{1}{2} v_i v^i \right) + \varepsilon(\rho) \right]$$

Notations

$$u_\mu \equiv \partial_\mu \theta + \alpha \partial_\mu \beta \,, \qquad \mu = 0, 1, 2; \quad i = 1, 2.$$

Relabeling symmetry: $\alpha \to \alpha + f'(\beta)$, $\theta \to \theta - f(\beta)$.

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Notations

$$u_\mu \equiv \partial_\mu \theta + \alpha \partial_\mu \beta\,, \qquad \mu = 0,1,2; \quad i = 1,2.$$

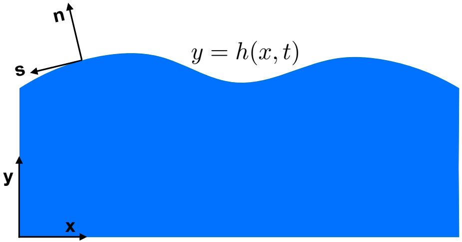
Variation over v^i gives: $\quad v_i = u_i = \partial_i \theta + \alpha \partial_i \beta, \quad i = 1,2.$

Variation over θ gives: $\quad \partial_t \rho + \partial_i(\rho v_i) = 0.$

Other variations plus algebra give Euler equation ($p \equiv \rho\varepsilon_\rho - \varepsilon$)

$$\partial_t v_i + (v_j \partial_j) v_i = -\rho^{-1} \partial_i p\,.$$

Introduction: Free surface boundary conditions



Kinematic boundary condition. Fluid particle on a surface Γ remains on a surface.

$$\left(\frac{\partial \Gamma}{\partial t} - \boldsymbol{v}\right)_n = 0 \qquad \text{or} \qquad \partial_t h + v_x \partial_x h = v_y \,.$$

Dynamical boundary conditions. Vanishing of stress forces on the boundary.

$$f_i = T_{ij} n_j \Big|_\Gamma = 0 \qquad \text{or} \qquad T_{nn} = 0\,, \; T_{sn} = 0\,. \qquad \text{two conditions}$$

Luke's variational principle

Consider the action defined for domain $y \le h(x,t)$:

$$S[\rho, \theta, \alpha, \beta, v_i] = -\int dt \int dx \int_{-\infty}^{h(x,t)} dy \left[\rho \left(u_0 + u_i v^i - \frac{1}{2} v_i v^i \right) + \varepsilon(\rho) \right]$$

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Using bulk equations of motion we obtain the following.

Variation over θ at the boundary gives the kinematic boundary condition

Variation over $h(x,t)$ gives: the dynamic boundary condition $p\Big|_{\Gamma} = 0$ for free surface

As $T_{ij} = -p\delta_{ij}$ the transverse dynamic b.c. is satisfied automatically.

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The action encodes both bulk hydro and free surface boundary conditions

Remarks on boundary conditions: Free surface

$$\partial_t \rho + \partial_i(\rho v_i) = 0\,, \qquad \left(\frac{\partial \Gamma}{\partial t} - \boldsymbol{v}\right)_n = 0$$

$$\partial_t p_i + \partial_j(p_i v_j) = \partial_j T_{ij}\,, \qquad T_{ij} n_j \Big|_\Gamma = 0$$

Constitutive relations. $a_i^* \equiv \epsilon_{ij} a_j$

$$p_i = \rho v_i\,, \qquad \tilde{p}_i = \rho v_i + s\partial_i^* \rho\,,$$

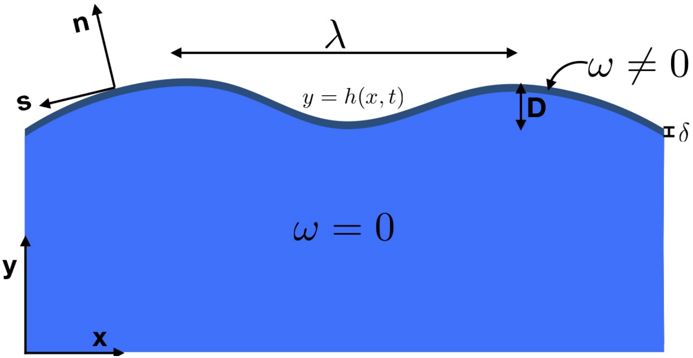
$$T_{ij} = -p\delta_{ij}\,, \qquad \tilde{T}_{ij} = -\left(p - \frac{s}{2}\rho\omega\right)\delta_{ij} - \frac{s}{2}\rho(\partial_i v_j^* + \partial_i^* v_j) - \frac{s}{2}\rho(\partial_k v_k)\epsilon_{ij}\,.$$

Bulk equations for ρ and v_i are identical for both cases but
dynamic boundary conditions are different!
The corresponding actions must differ by boundary terms.

Boundary vorticity layer

- In the presence of viscosity the solution of surface wave problem cannot be potential everywhere!
- Tangent stress at the boundary due to the boundary motion results in the vorticity at the boundary.
- Oscillating boundary layer forms (similar to Lamb '32, without Hall viscosity).
- Hall viscosity modifies the structure of boundary layer.
- Our goal is to write equations for effective dynamics of the boundary assuming that the boundary layer is very thin in the incompressible limit $c_s \to \infty$.

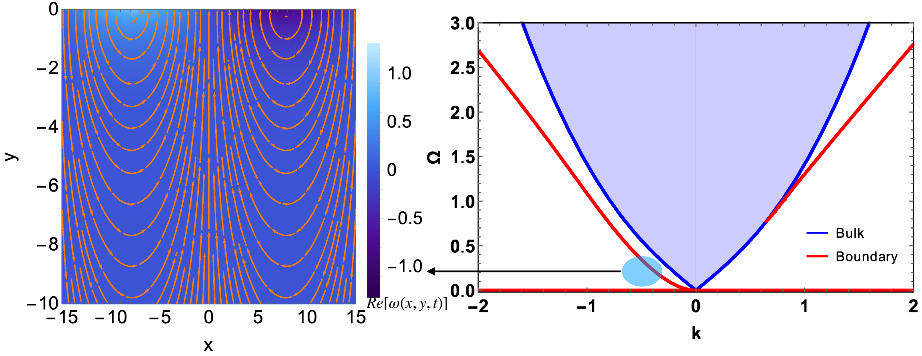
Boundary vorticity layer



$$\nu_o/c_s \sim \delta \ll D \ll \lambda\,, \qquad \omega \sim D\Omega c_s/\nu_o, \qquad \omega\delta \sim D\Omega$$

AA, T. Can, S. Ganeshan, G. Monteiro, arXiv:1907.11196

Linear odd surface waves



For small k: $\Omega = -2\nu_o k|k|$.

Chiral Burgers Equation

Weakly nonlinear surface waves for incompressible fluid with odd viscosity are described by *Chiral Burgers Equation* AA, T. Can, S. Ganeshan, SciPost Phys. **5**, 010 (2018)

$$u_t + 2uu_x - 2i\nu_o u_{xx} = 0$$

$u(x)$ is a complex function, $u(x+iy)$ is analytic in the lower half-plane $\mathrm{Im}\,(z) = y \le 0$.

Relation to original variables

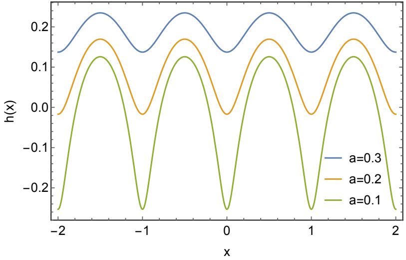
$$u = v_x + iv_y \;\Big|_{\Gamma}$$

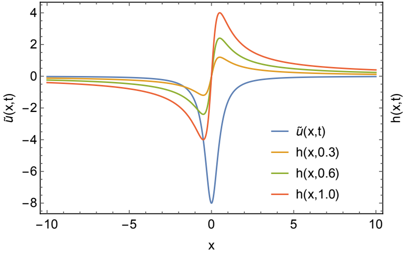
Dobrokhotov, Maslov, Tsvetkov, '92; Senouf, Caflisch, Ercolani, '96
Cf. Kuznetsov, Spector, Zakharov, '94

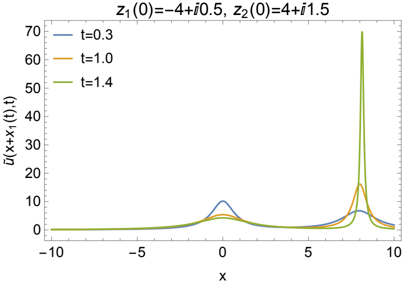
Applications

chiral wave (left moving)
propagates in the absence of gravity
in linear regime $\Omega = -2\nu_o k^2$

Cf. Lushnikov, '04







Bulk

Parity breaking term, first order in gradients

$$S_{\mathcal{M}}[\rho,\theta,\alpha,\beta,v_i] = -\int dt \int_{\mathcal{M}} d^2x \left[\rho\left(u_0 + u_i v^i - \tfrac{1}{2}v_i^2\right) + \varepsilon(\rho) - \nu_o v_i \partial_i^* \rho\right]$$

where $$u_\mu \equiv \partial_\mu\theta + \alpha\partial_\mu\beta\,,$$

Variation over v_i gives $$v_i = u_i - \nu_o \partial_i^* \ln\rho.$$

The action produces equations for $\theta, \rho, \alpha, \beta$ which imply hydrodynamics with

$$T_{ij} = -p\delta_{ij} + \nu_o\rho(\partial_i v_j^* + \partial_i^* v_j)\,.$$

The extra term in the action is equivalent (up to boundary terms) to $\nu_o\rho\,\omega$.
Gives "wrong" boundary conditions, e.g., $v_{tangent} = 0$.

Boundary

We consider domain $\mathcal{M}: \ y \leq h(x,t)$ with boundary $\Gamma: \ y = h(x,t)$

The boundary action:

$$S_\Gamma[\tilde{\rho}, \phi, h] = -\nu_o \int dt \int dx \left[\tilde{\rho}\, h_x h_t + \phi_x \phi_t - 2\phi_t \sqrt{\tilde{\rho}(1+h_x^2)}\right]$$

with $\tilde{\rho}(x,t) = \rho(x, h(x,t); t)$ produces correct dynamic boundary conditions for the fluid with ν_o.

Remark: ϕ-field is needed to make the action boundary reparametrization invariant. It can be integrated out for the price of non-locality.

Main result: Hydro action with Hall viscosity

The action

$$S = S_{\mathcal{M}} + S_{\Gamma}$$

$$S_{\mathcal{M}}[\rho, \theta, \alpha, \beta, v_i] = -\int dt \int_{\mathcal{M}} d^2x \left[\rho \left(u_0 + u_i v^i - \tfrac{1}{2} v_i^2\right) + \varepsilon(\rho) - \nu_o v_i \partial_i^* \rho\right]$$

$$S_{\Gamma}[\tilde{\rho}, \phi, h] = -\nu_o \int dt \int dx \left[\tilde{\rho}\, h_x h_t + \phi_x \phi_t - 2\phi_t \sqrt{\tilde{\rho}(1 + h_x^2)}\right]$$

gives both bulk hydrodynamic equations and free surface boundary conditions for two-dimensional compressible fluid with odd viscosity.

Remark: Adding temperature or external gauge field to the action is rather straightforward.

AA, T. Can, S. Ganeshan, G. Monteiro, arXiv:1907.11196

From action to Poisson structure (bulk)

It is straightforward to derive Hamiltonian structure from the action. The (bulk) symplectic part ρu_0 is conventional and we obtain for ρ and u_i standard PBs.

$$\{\rho, \rho'\} = 0\,, \qquad \{\rho, u_i'\} = \partial_i \delta(x - x')\,, \qquad \{u_i, u_j'\} = -\frac{\partial_i u_j - \partial_j u_i}{\rho}\delta(x - x')\,.$$

$$v_i = u_i - \nu_o \partial_i^* \ln \rho \quad \longrightarrow \quad \text{modified brackets for } \rho \text{ and } v_i.$$

Immediate consequence:

$$I_n = \int dx\, \rho \left(\frac{\omega_u}{\rho}\right)^n = \int dx\, \rho \left(\frac{\omega + \nu_o \Delta \ln \rho}{\rho}\right)^n\,, \qquad n = 0, 1, 2, \ldots$$

are Casimirs of hydro with odd viscosity.

Remark: in external magnetic field $\omega \to \omega + B$.

From action to Hamiltonian

Remarkably the Hamiltonian for the constructed action is the same:

$$H = \int d^2x \left[\frac{\rho v^2}{2} + \varepsilon(\rho) \right] .$$

Only Poisson structure changed!

Few remarks:

- Similar situation occurs in the presence of magnetic field or system's rotation. Odd viscosity is a higher gradient analogue of those.
- Reminds topological terms which do not change stress-energy tensor. However, the momentum density did change implying non-trivial coupling to time components of the metric.
- Similarly to topological terms odd viscosity does change boundary dynamics in a profound way.

Conclusions

1. *Chiral Burgers equation* describing nonlinear boundary dynamics in incompressible limit is derived for fluid with odd viscosity
2. Few exact solutions of the *chiral Burgers equation* are obtained
3. Variational principle for incompressible fluid with Hall viscosity and free surface is constructed
4. A non-trivial boundary term is needed to give correct boundary conditions
5. The odd viscosity terms modify conventional Poisson's brackets in Hamiltonian formulation.

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